A Pictorial Explanation of Stellar Aberration

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Stellar aberration is the phenomenon whereby the observed angular position of a star depends on the relative motion between the star and Earth. Specifically, a telescope must be tilted slightly into the direction of motion of Earth relative to the star.[1] There are in fact three different angular positions of interest: the observed position of the star from Earth, the actual position of the star (at the instant of observation) as measured using Earth's clocks and rulers, and the actual position of the star (relative to Earth) as measured using the star's clocks and rulers. Clear diagrams show that none of these three angular positions are in general equal to each other, and help explain why the effect in practice depends only on Earth's motion and not on the star's motion, in apparent violation of the relativity of motion.

There are at least four applications of stellar aberration. First, Ferguson[2] comments that stellar aberration is the oldest direct proof of Earth's motion about the Sun. Second, Bradley's measurements of the observed angular deviation in arc seconds for eight stars in 1727 agree with modern values to within 1% and gave the most precise value for the speed of light obtained up to that time. (Reference 3 provides a readable overview of who Bradley was, what motivated his measurements, and the details of his telescope and what he saw when he looked through it.) Third, the agreement between experiment and theory for stellar aberration established that if there were an ether, Earth would have to be moving relative to it, directly confronting the null result of the Michelson-Morley experiment, which at the time suggested that the ether gets dragged along with Earth (in the way that the atmosphere is), as is discussed in some introductory physics textbooks.[4] Fourth, stellar aberration continues to be used today as an accurate method of determining Earth's orbital velocity about the Sun. Additional aberration due to Earth's rotation on its axis is also measurable, and the European GAIA (Global Astrometric Interferometer for Astrophysics) satellite may even have enough accuracy to measure the orbit of our solar system around the center of our galaxy.[5] Entire chapters are devoted to the topic of stellar aberration in books on astronomy[6,7] and special relativity.[8] The topic could be tied into an introductory discussion of relative velocity, historical astronomy, or special relativity.

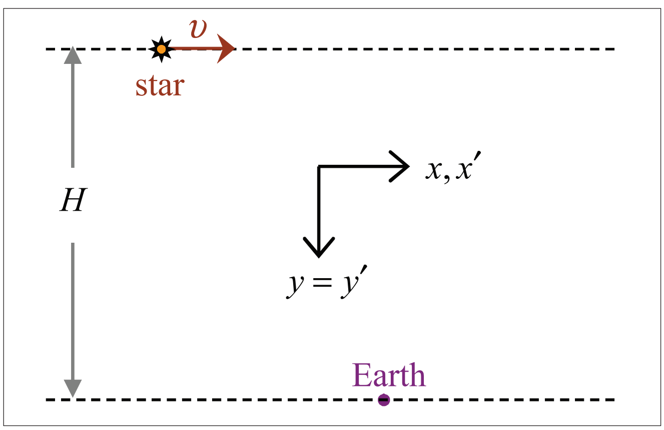

Fig. 1. A star is moving to the right at speed *v* relative to Earth along a line of motion that is a vertical distance *H* away from Earth. The stellar frame makes measurements in terms of unprimed position and time coordinates (*x*, *y*, *t*) while the terrestrial frame uses primed coordinates (*x*′, *y*′, *t*′) with the positive spatial directions as shown. At some instant in time used to define both *t* = 0 and *t*′ = 0, the star emits a brief flash of light isotropically in all directions. The star's position at that instant defines *x* = 0, *x*′ = 0, and *y* = *y*′ = 0.

Suppose Earth is moving leftward at speed v relative to a star, or equivalently the star is moving rightward at speed v relative to Earth, as depicted in Fig. 1. If the velocities of both the star and Earth remained constant, the distance of closest approach between them would be H. For the purposes of making measurements of position x and y and of time t of an event occurring anywhere in the plane of the page, imagine that an observer named Alice is on the surface of the star. (For simplicity, assume that neither the star nor Earth rotate on their axes.) Likewise, an observer named Carlos is on the surface of Earth and can make measurements of position x' and y' and time t' of the same event. The zeroes of time and position, and the positive directions of the spatial axes are explained in the caption of Fig. 1. Since the relative motion of the stellar and terrestrial frames is purely horizontal in that figure, both observers agree on measurements of the transverse spatial coordinate so that $y = y'$. Thus, from now on, reference will only be made to y and not to y'.

The star emits a brief isotropic flash of light at the initial instant ($t = 0 = t'$) that propagates away as a circular wavefront in the plane of the figure. In order to better track the motion of the wavefront, it is helpful to introduce another observer called Bob and a set of four detectors numbered 1 through 4, as sketched in Fig. 2. Detectors 1 and 2 are connected rigidly to Alice in the positive and negative x directions, respectively, at a *proper* distance of L away from her. Likewise detectors 3 and 4 are connected rigidly to Bob in the positive and negative x' directions, respectively, at a *proper* distance of L away from him. Figure 2 shows the stellar and terrestrial frames as enclosing rectangles, but it is to be understood that both frames actually extend infinitely far in all four compass directions. Thus, it might be best to think of the two frames as being two planes parallel to the page, with one plane slightly displaced perpendicularly out of the page to avoid interference between them, like two independent *Flatland* worlds in the spirit of Abbott's classic novel about life in a two-dimensional universe.[9] For drawing purposes, Alice and Bob are shown in Fig. 2 as being shifted vertically away

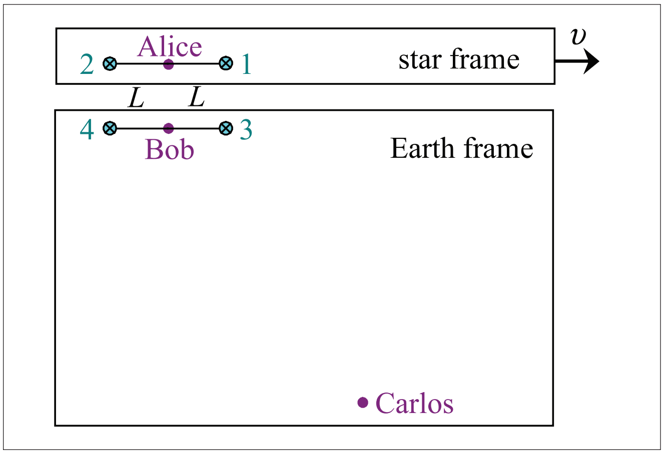

Fig. 2. Alice is on the star moving rightward at speed *v* relative to Carlos on Earth. Bob is stationary relative to Earth and happens to be located at the origin $x = 0 = x'$ and $y = 0$ at the instant $t = 0 = t'$ that the star flashes. Both Alice and Bob have a pair of detectors rigidly connected rightward and leftward of their locations a distance *L* away from themselves *as measured in their own rest frames*. (Due to length contraction, that implies each will measure the other's observer-detector separation distance to be shortened to L/γ, where γ is the relativistic factor.)

from each other (and in subsequent figures the star flash will be shown as emanating from a point midway between them), but those two observers and the flash all actually coincide spatially at the initial instant.

Now consider observations of the propagating wavefront of light in the stellar frame of reference. Figure 3 shows snapshots at three consecutive times t. (To avoid cluttering the figure, the distances L to the four detectors and the numeric labeling of the detectors are suppressed, because they are the same as in Fig. 2.) In each panel, the current location of the wavefront is indicated by a solid circular arc and its previous locations by dashed circular arcs. The wavefront at the initial instant in panel (a) is actually of negligible diameter (since Alice and Bob are then both at the origin at the star) but is shown as having a finite size for clarity. Since Alice observes light to propagate at speed c in all directions (regardless of the motion of her frame, according to special relativity), the wavefronts in all three panels are circles centered on her. The star is only shown in panel (a) because it is emitting light at $t = 0$; in panels (b) and (c) it is no longer emitting and is thus no longer visible to observers. In panel (b), the wavefront strikes Alice's two detectors *simultaneously*, because she is equally far away from both of them. However, at this time T_1 the wavefront has already passed Bob's right detector 3 but not yet reached his left detector 4. In panel (c), a green ray of light is drawn perpendicular to the wavefront at the point that it strikes Earth. From the diagram, one sees that the angle θ this ray makes relative to the direction of relative motion of Earth is equal to the *actual angular position of the star relative to Earth at the instant of observation in the stellar reference frame.*

Next consider observations of the wavefront in the terrestrial frame of reference. Figure 4 shows snapshots at three consecutive times t'. Since Bob observes light to propagate at speed c in all directions (regardless of the motion of his frame,

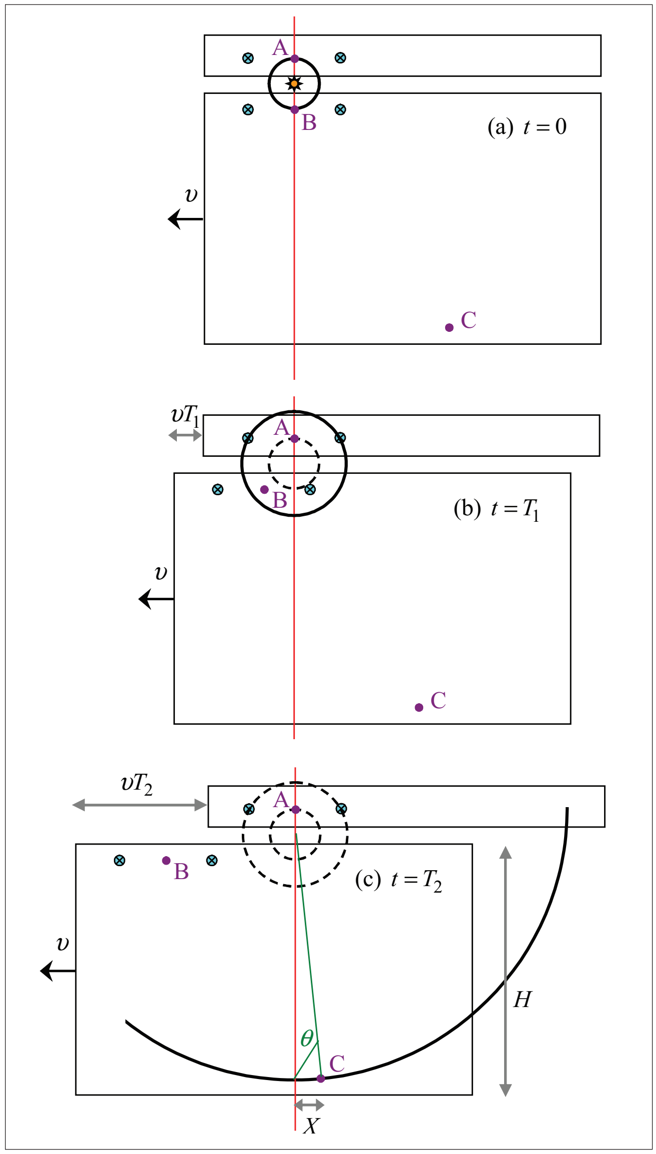

Fig. 3. Snapshots of the circular wavefront at three instants of time *t* as measured in the (unprimed) stellar frame: (a) at the initial instant of the flash; (b) at the instant T_1 that the wavefront simultaneously strikes Alice's detectors 1 and 2; and (c) at the instant T_2 that the wavefront reaches Earth at position $x = X$ and $y = H$. In each panel, the current location of the wavefront is indicated by a solid circular arc and its previous locations by dashed circular arcs. The red line is fixed on Alice's horizontal position at $x = 0$. The leftward displacements of Bob (B) and Carlos (C) relative to Alice (A) are vT_1 and vT_2 at times T_1 and T_2, respectively.

according to special relativity), the wavefronts in all three panels are circles centered on him.[10] Again the star is only shown in panel (a), while it is emitting light at $t' = 0$, and not in panels (b) and (c). In panel (b), the wavefront strikes Bob's two detectors *simultaneously,* because he is equally far away from both of them. However, at this time T_1' the wavefront has already passed Alice's left detector 2 but not yet reached her right detector 1. Comparing Figs. 3 and 4, one discovers that $T_1 = L/c = T_1'$ and yet simultaneous events (namely the arrival of the light wavefront at two detectors) in one frame are not observed as simultaneous events in the other frame,

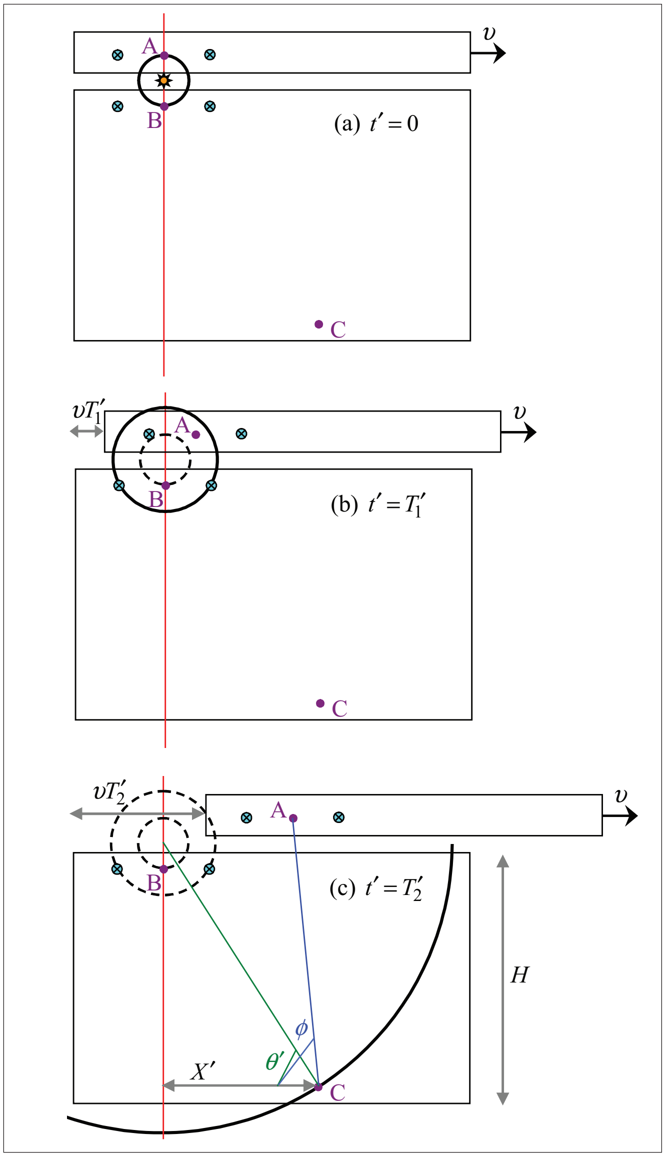

Fig. 4. Snapshots of the circular wavefront at three instants of time t' as measured in the (primed) terrestrial frame: (a) at the initial instant of the flash; (b) at the instant T_1' that the wavefront simultaneously strikes Bob's detectors 3 and 4; and (c) at the instant T_2' that the wavefront reaches Earth at position $x' = X'$ and $y = H$. In each panel, the current location of the wavefront is indicated by a solid circular arc and its previous locations by dashed circular arcs. The red line is fixed on Bob's horizontal position at $x' = 0$. The rightward displacement of Alice (A) relative to Bob (B) and Carlos (C) are vT_1' and vT_2' at times T_1' and T_2', respectively.

which illustrates the *relativity of simultaneity.*[11] In panel (c) of Fig. 4, a green ray of light is drawn perpendicular to the wavefront at the point that it strikes Earth. The angle θ' that this ray makes relative to the direction of relative motion of Earth is the *apparent angular position of the star relative to Earth at the instant of observation in the terrestrial reference frame.* It is the angle at which a telescope on Earth must be oriented to observe the star. Comparing Figs. 3 and 4, one sees that $\theta' < \theta$. The fact that these two angles are different is called *stellar aberration.*[12] In the online appendix,[13] mathematical relationships between the angles are derived. Also shown in Fig. 4(c) is a blue line connecting Carlos on Earth to Alice on the star. The angle ϕ that this line makes relative to the direction of relative motion of Earth is the *actual angular position of the star relative to Earth at the instant of observation in the terrestrial reference frame*. The angles θ in Fig. 3(c) and ϕ in Fig. 4(c) are *not* equal because[14] $T_2 \neq T_2'$ (in contrast to $T_1 = T_1'$). If one compares θ' to ϕ so that both angles are measured in the terrestrial frame, thereby avoiding the relativistic factor γ when jumping into the stellar frame to compare θ' to θ instead, then the *relativistic* stellar aberration formula reduces to the *classical* Bradley formula, as also shown in the online appendix. The upshot is that *a telescope needs to be tilted* away from the star's actual angular position, indicated by the blue line in Fig. 4(c), *into the direction of Earth's leftward velocity relative to the star* to obtain the green line in the same figure. This classical aberration is simply the result of the time delay between the emission of the flash by the star when it was at Bob's location and the observation of the light by Carlos; during that propagation time T_2', the star moved to Alice's current location in Fig. 4(c).

Finally, keeping in mind that aberration is entirely (in the classical case) or largely (in the relativistic case) due to the delay between emission and observation of the starlight, we can now understand why in practice only the motion of Earth matters and not the motion of the star, despite the fact that everything presented thus far is completely symmetric between the motion of both bodies (as special relativity insists it must be). In fact, textbooks typically discuss aberration by *only* considering the motion of the telescope and pass over in silence the question of what happens if the *star* is moving, hoping perhaps that students will not notice the omission. But a more complete explanation is not difficult to provide. First, consider a star that is moving at constant velocity. Then its angular position will be aberrated as Fig. 4 shows. However, since Earthlings have no way of measuring the star's current actual position[15] (and no information about it can travel to Earth faster than c), that fact has no *observable* consequences. Second, consider what happens if the star's velocity is not constant. An interesting example is that of binary stars revolving around a common center of mass.[16] For simplicity, assume their center of mass is at rest relative to Earth and is fixed at point B in Fig. 4. If at some instant one star in the binary is moving to the right at high speeds and the other is moving to the left at high speeds, does a telescope need to be flung between a leftward and a rightward orientation to see both of them? Certainly not! Since each star is revolving around point B, the telescope simply needs to be aimed at that point (assuming it has a wide enough field of view to encompass both stars near that location). Each star repeatedly U-turns around and so its actual location is always near B and it never travels to point A in Fig. 4(c). To put it another way, in astronomy we are always looking back in time when we observe distant stars. We see the star where it used to be and not where it currently is. But for binary stars orbiting around point B, that means they will always be observed near B.

In contrast, consider what happens to Earth as it revolves around the Sun at a speed[17] of $v = 2\pi(1\ \text{AU})/(1\ \text{y}) = 30$ km/s.

For simplicity, imagine a star that is at rest in the sky relative to the Sun and is located perpendicular to the plane of the ecliptic along its polar axis. In some months of the year, we will be traveling leftward around the Sun, as in Fig. 4. At those times, we will have to incline our telescopes leftward of a star's actual position by $20.5''$ (as shown in the online appendix) in order to view it. Twenty seconds of arc is small but measurable, approximately the angle subtended by an American dime at a distance of 200 m. Bradley's telescope of 1727 could measure angular orientations down to a remarkable quarter second of arc.[3] But six months later, when we are traveling rightward, we will have to tilt the telescope rightward by $20.5''$. Thus, there will be a detectable annual *variation* in the angular orientation of the telescope, not to be confused with parallax,[1] which typically is masked because it has a much smaller amplitude (and a different phase) than the aberration. To conclude, the reason that aberration is in practice asymmetric with respect to motion of the star and Earth is not because the effect itself is *conceptually* asymmetric but because its experimental *observation* requires such a lack of symmetry. That is, stellar aberration results from a transformation between the reference frames of two different observers (viz. Earth at two different points in its orbit) rather than between a single observer and source.[18]

References

1. T. K. Timberlake, "Seeing Earth's orbit in the stars: Parallax and aberration," *Phys. Teach.* **51**, 478–481 (Nov. 2013).
2. D. C. Ferguson, "Stellar aberration and apparent rotation: A direct link," *Am. J. Phys*. **39**, 1089–1090 (Sept. 1971).
3. A. B. Stewart, "The discovery of stellar aberration," *Sci. Am.* **210**, 100–109 (March 1964).
4. R. Wolfson, *Essential University Physics*, 3rd ed. (Pearson, San Francisco, 2016), p. 624.
5. G. Preti and F. de Felice, "Relativistic satellite astrometry at microarcsecond precision and the measurement of the stellar aberration," *A&A* **513**, A68 (April 2010).
6. J. Kovalevsky and P. K. Seidelmann, *Fundamentals of Astrometry* (Cambridge Univ. Press, UK, 2004), Chap. 6.
7. E. W. Woolard and G. M. Clemence, *Spherical Astronomy* (Academic Press, New York, 1966), Chap. 6.
8. A. I. Miller, *Albert Einstein's Special Theory of Relativity* (Addison-Wesley, Reading, MA, 1981), Chap. 10.
9. E. A. Abbott, *Flatland: A Romance of Many Dimensions* (Dover, NY, 1992).
10. Students sometimes ask how the circular wavefronts can be centered on *both* Alice and Bob (as seen by each of them). Which figure, 3 or 4, depicts what is "really" happening? This is the time to remind students that in special relativity (and quantum mechanics) there is no *absolute* reality. There is only what a *particular* experiment measures *relative* to a particular reference frame.
11. R. Harris, *Modern Physics*, 2nd ed. (Pearson, San Francisco, 2008), p. 9.
12. A. P. French, *Special Relativity* (Norton, New York, 1968), p. 132.
13. The appendix can be found at *TPT Online* at http://dx.doi.org/10.1119/1.5126831 under the Supplemental tab.
14. For the two green rays of light in Figs. 3(c) and 4(c), the geometry implies that the invariant speed of light is $c = (X^2 + H^2)^{1/2} / T_2 = (X'^2 + H^2)^{1/2} / T'_2$. Since $X \neq X'$ it follows that $T_2 \neq T_2'$. Also see Z. Mulaj and P. Dhoquina, "Pythagoras theorem and relativistic kinematics," *AIP Conf. Proc*. **1203**, 1453–1455 (2010).
15. K. Kassner, "Why the Bradley aberration cannot be used to measure absolute speeds: A comment," *Europhys. Lett*. **58**, 637–638 (May 2002).
16. T. E. Phipps Jr., "Relativity and aberration," *Am. J. Phys*. **57**, 549–551 (June 1989).
17. The Sun is located one astronomical unit away ($1\ \text{AU} = 1.5\times 10^8$ km) and it takes Earth one year ($1\ \text{y} = \pi\times 10^7$ s) to complete an orbit.
18. E. Eisner, "Aberration of light from binary stars: A paradox?" *Am. J. Phys*. **35**, 817–819 (Sept. 1967).

Carl Mungan *is currently teaching modern physics to physics and nuclear engineering majors, which motivated him to try to better explain stellar aberration than the typical brief mention made in textbooks.*
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Online Appendix for "A Pictorial Explanation of Stellar Aberration"

The following two sections provide quantitative formulae for the effects discussed in the main text.

Mathematical relationship between the angles θ, θ', and ϕ

Noting that the primed (terrestrial) frame in the figures moves leftward at speed υ relative to the unprimed (stellar) frame, while the $+x$ and $+x'$ axes point rightward, the Lorentz transformation implies that

$$x' = \gamma(x + \upsilon t) \quad \text{where} \quad \gamma \equiv 1\Big/\sqrt{1-\upsilon^2/c^2}\,. \tag{1}$$

For the arrival of the wavefront on Earth in Figs. 3(c) and 4(c), Eq. (1) becomes

$$X' = \gamma(X + \upsilon T_2)\,. \tag{2}$$

Applying this result in Fig. 4(c), one sees that

$$\tan\theta' = \frac{H}{X'} = \frac{H}{\gamma(X + \upsilon T_2)}\,, \tag{3}$$

whereas the geometry in Fig. 3(c) leads to both

$$\tan\theta = \frac{H}{X} \tag{4}$$

and

$$\cos\theta = \frac{X / T_2}{c} \tag{5}$$

because the numerator of this last ratio equals the x-component of the velocity of the green ray of light in the stellar frame. Finally, divide the numerator and denominator of Eq. (3) by X to obtain

$$\tan\theta' = \frac{\tan\theta}{\gamma(1+\beta\sec\theta)} \quad \text{where} \quad \beta \equiv \upsilon / c \tag{6}$$

using Eqs. (4) and (5). Equation (6) is one of the standard ways of writing the relativistic stellar aberration formula, as shown in the next section. Assuming $0° < \theta < 90°$ and $\upsilon > 0$, this formula implies that $\theta' < \theta$ as in the figures. Alternatively, Eq. (6) can be inverted[1] to give

$$\tan\theta = \frac{\tan\theta'}{\gamma(1-\beta\sec\theta')}\,. \tag{7}$$

To find ϕ, observe that point A in Fig. 4(c) is shifted horizontally a distance $\upsilon T_2'$ to the right of point B. Therefore

$$\tan\phi = \frac{H}{X' - \upsilon T_2'}\,. \tag{8}$$

However the inverse Lorentz transformation[1] implies that Eq. (2) becomes

$$X = \gamma(X' - \upsilon T_2'). \tag{9}$$

Substituting this result into Eq. (8) and comparing the result to Eq. (4) results in

$$\tan\phi = \gamma \tan\theta \tag{10}$$

which implies that $\phi > \theta$ if the Earth and star are in motion relative to each other. We see that ϕ and θ are connected by the relativistic factor γ because of the transformation[2] from the stellar to the terrestrial frames of reference between Figs. 3 and 4.

There is another informative way to relate the angles to each other. Figure 4(c) shows that

$$\cos\theta' = \frac{X'/T_2'}{c} \tag{11}$$

in analogy to Eq. (5). Dividing the numerator and denominator of Eq. (8) by X' leads to

$$\tan\phi = \frac{\tan\theta'}{1 - \beta\sec\theta'} \tag{12}$$

using Eqs. (3) and (11). This result is the classical (Bradley) stellar aberration formula because the relativistic (Lorentz) transformation was *not* used anywhere in its derivation. To put it another way, classically one sets $\gamma = 1$, in which case Eq. (10) becomes $\phi = \theta$ and Eq. (7) becomes Eq. (12). It is exactly the formula one would obtain by treating the photons like raindrops falling at speed c and angle ϕ in a ground frame [following the blue trajectory in Fig. 4(c)] but observed by the driver of a car moving leftward at speed υ to fall at angle θ' [following the green trajectory in Fig. 4(c)]. From the driver's point of view, it is the raincloud that is moving rightward at speed υ. That would give the raindrops an additional rightward velocity component of υ so that they appear to fall at a shallower angle to the driver than they would to a stationary observer.[3]

The equations presented here can also be used in the treatment of the relativistic Doppler shift for a source and receiver moving at an angle with respect to each other.[4] Equations (6) and (7) give the transformation between the angles θ and θ' of the emitted electromagnetic ray measured in the frames of the source and receiver. Consequently an instructor who takes the time to carefully explain aberration will find that it pays off in subsequent developments in special relativity.

Derivation of the relativistic stellar aberration formula from Lorentz velocity addition

Equation (1) gives the spatial Lorentz transformation. The corresponding temporal transformation is found by exchanging[5] $x \leftrightarrow ct$ to obtain

$$ct' = \gamma(ct + \upsilon x/c) \quad \Rightarrow \quad t' = \gamma(t + \upsilon x/c^2). \tag{13}$$

Take the ratio of the differentials of Eqs. (1) and (13) to get the x'-component of the velocity of an object (in our case of a photon) as

$$u_x' \equiv \frac{dx'}{dt'} = \frac{\gamma(dx + \upsilon dt)}{\gamma(dt + \upsilon dx/c^2)}. \tag{14}$$

Finally divide every term in the numerator and denominator by dt to end up with

$$u'_x = \frac{u_x + \upsilon}{1 + \upsilon u_x / c^2} \tag{15}$$

which can be conveniently remembered as being the "sum divided by unity plus the dimensionless product" of the two "other" relevant velocities along the *x* direction. Likewise for the *y*-components one obtains

$$u'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma(dt + \upsilon dx / c^2)} = \frac{u_y}{\gamma(1 + \upsilon u_x / c^2)}. \tag{16}$$

Equations (15) and (16) are the Lorentz velocity addition formulae. Using them, the angle θ' of the light ray in the (primed) terrestrial frame is found from

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma(u_x + \upsilon)}. \tag{17}$$

Dividing every term in the numerator and denominator by u_x and noting that Eq. (5) can be rewritten as $\cos\theta = u_x / c$, the final result is

$$\tan\theta' = \frac{\tan\theta}{\gamma(1 + \beta\sec\theta)} \tag{18}$$

in agreement with Eq. (6).

For the case of Earth's motion around the sun, $\upsilon = 2\pi R / T$ where *R* is one astronomical unit and *T* is one year. Equation (18) can then be used to calculate the angular change in a telescope's orientation $\theta - \theta'$ as plotted in Fig. A1. Defining $\delta \equiv 90° - \theta$ and $\delta' \equiv 90° - \theta'$, Eq. (18) can be approximated as

$$\frac{\cos\delta'}{\sin\delta'} \approx \frac{\cos\delta}{\sin\delta + \beta} \tag{19}$$

since $\upsilon << c$. One sees from Fig. A1 that the largest aberration occurs as θ and θ' approach 90° which implies that δ and δ' approach 0°. In that limit, Eq. (19) becomes

$$\delta' \approx \delta + \beta \quad \Rightarrow \quad (\theta - \theta')_{\max} = \beta \tag{20}$$

which is called the aberration constant[6] and is equal to $20.5''$ for Earth's orbital speed of 29.8 km/s.

References

1. To invert any transformation between the primed and unprimed frames, exchange primed and unprimed variables ($x \leftrightarrow x'$, $t \leftrightarrow t'$, and likewise for *X*, *θ*, T_2, and the components of *u*) and reverse the sign of every υ.
2. A. Gjurchinovski, "Relativistic aberration of light as a corollary of the relativity of simultaneity," *Eur. J. Phys.* **27**, 703–708 (July 2006).
3. A practical consequence of this raindrop aberration is the Poynting-Robertson effect (https://www.revolvy.com/page/Poynting-Robertson-effect) whereby radially directed

photons from the sun strike dust particles in circular orbit with a backward-directed tangential component that cause the particles to spiral into the sun.

4. S.T. Thornton and A. Rex, *Modern Physics for Scientists and Engineers*, 4th ed. (Brooks/Cole, Boston, 2013), p. 57. Specifically, the connection between stellar aberration and the relativistic Doppler shift can be made more apparent by rewriting (see the attached derivation) Eq. (6) as $\tan(\theta'/2) = \tan(\theta/2)\sqrt{(1-\beta)/(1+\beta)}$.
5. Time and space are both "dimensions" to be treated on an equal footing in special relativity when both are expressed in units of length by multiplying time by the speed of light.
6. F.J. Shore, "Stellar aberration, invariant velocities, and Earth's hodograph," *Am. J. Phys.* **57**, 948–949 (Oct. 1989).

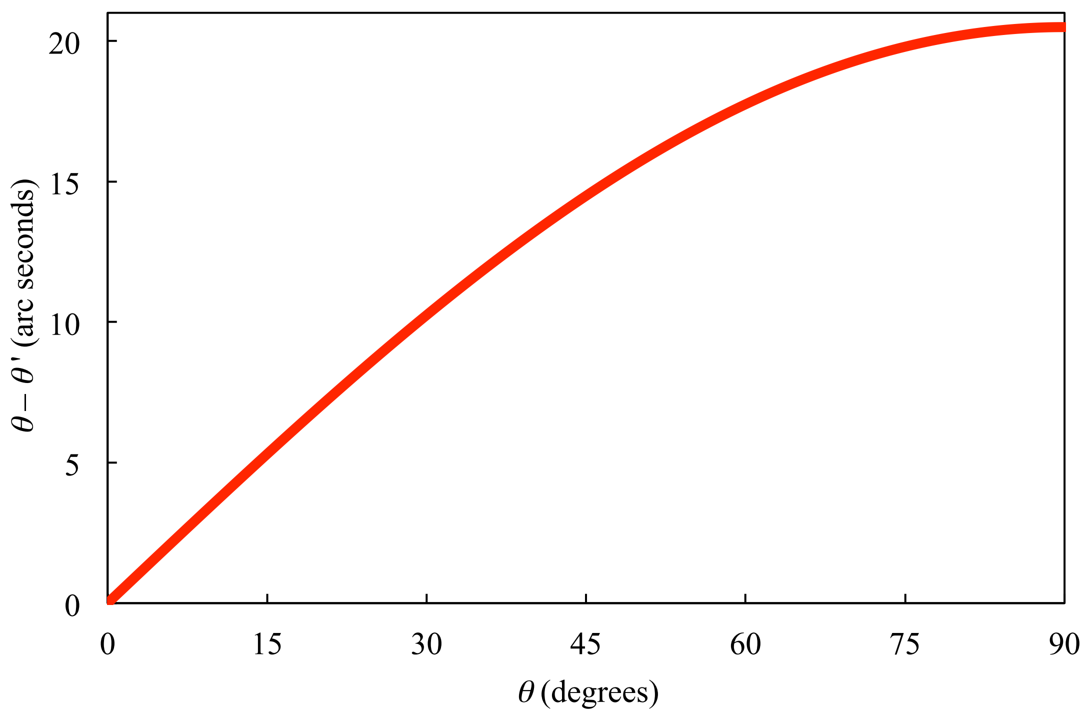

Figure A1. Angular aberration as a function of the actual angular position θ of a star relative to the direction of Earth's orbital velocity in the stellar reference frame. Thus a telescope would have to be reoriented by 41 seconds of arc if a star located along the axis perpendicular to the ecliptic were viewed six months apart.

Derivation of the Rewritten Form of Eq. (6) in Endnote 4 of the Online Appendix

Start from the inverted form of Eq. (6) given in Eq. (7) and multiply the numerator and denominator by $\cos\theta'$ to obtain

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' - \beta)} . \tag{A1}$$

Now substitute this result into the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$ to get

$$\frac{1}{\cos^2\theta} = \frac{\gamma^2(\cos\theta' - \beta)^2 + \sin^2\theta'}{\gamma^2(\cos\theta' - \beta)^2} \tag{A2}$$

whose reciprocal square root is

$$\cos\theta = \frac{\cos\theta' - \beta}{\sqrt{(\cos\theta' - \beta)^2 + (1 - \beta^2)\sin^2\theta'}} \tag{A3}$$

after dividing the numerator and denominator by γ. The terms under the square root sign in Eq. (A3) can be expanded and simplified using the identity $\sin^2\theta' + \cos^2\theta' = 1$ to end up with

$$\cos\theta = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta'} . \tag{A4}$$

As a result, one has

$$1 + \cos\theta = \frac{1 - \beta\cos\theta' + \cos\theta' - \beta}{1 - \beta\cos\theta'} \quad \text{and} \quad 1 - \cos\theta = \frac{1 - \beta\cos\theta' - \cos\theta' + \beta}{1 - \beta\cos\theta'} . \tag{A5}$$

Finally substitute these into the trigonometric identity

$$\tan\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \sqrt{\frac{(1 - \cos\theta')(1 + \beta)}{(1 + \cos\theta')(1 - \beta)}} = \sqrt{\frac{1 + \beta}{1 - \beta}} \tan\frac{\theta'}{2} \tag{A6}$$

valid for first-quadrant angles as in this article.