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INVESTIGATIONS OF AN EARLY SUMERIAN DIVISION PROBLEM, c. 2500 B.C.

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SUMMARY

Two Sumerian school tablets from c. 2500 B.C.--containing respectively a correct and an erroneous solution of the same problem of dividing a very large amount of grain into rations of 7 sila *each--are analyzed. It is suggested that the error committed cannot reasonably be explained by two earlier conjectures on the method of solution (normal "long division," and multiplication by the reciprocal), but only by conversion to an intermediary unit and calculation in two steps analogous to the principle of "long division." Also discussed are some possible implications of this result for contemporary Sumerian arithmetical abilities and general techniques.*

Nous analysons deux tablettes d'école sumériennes gravées environ 2500 ans avant notre ère. L'une est une solution correcte et l'autre, une solution erronée d'un même problème, à savoir la division d'une très grande quantité de grain en portions de 7 sila*. On montre que les deux méthodes de solution proposées jusqu'à maintenant (l'une analogue à la méthode moderne, l'autre étant la multiplication par le nombre inverse) ne suffisent pas à expliquer l'erreur commise, et qu'il faut supposer une conversion en une unité intermédiaire suivie d'un processus en deux étapes de "division de nombre complexe". Nous discutons des implications de cette analyse sur notre connaissance du savior et des pratiques arithmétiques contemporains des sumériens.*

In an important paper on the prehistory of Babylonian mathematics, Marvin A. Powell [1976] discussed two tablets from Fara (ancient Šuruppak) from c. 2500 B.C., which he identified as two school exercises (one correct and one erroneous) dealing

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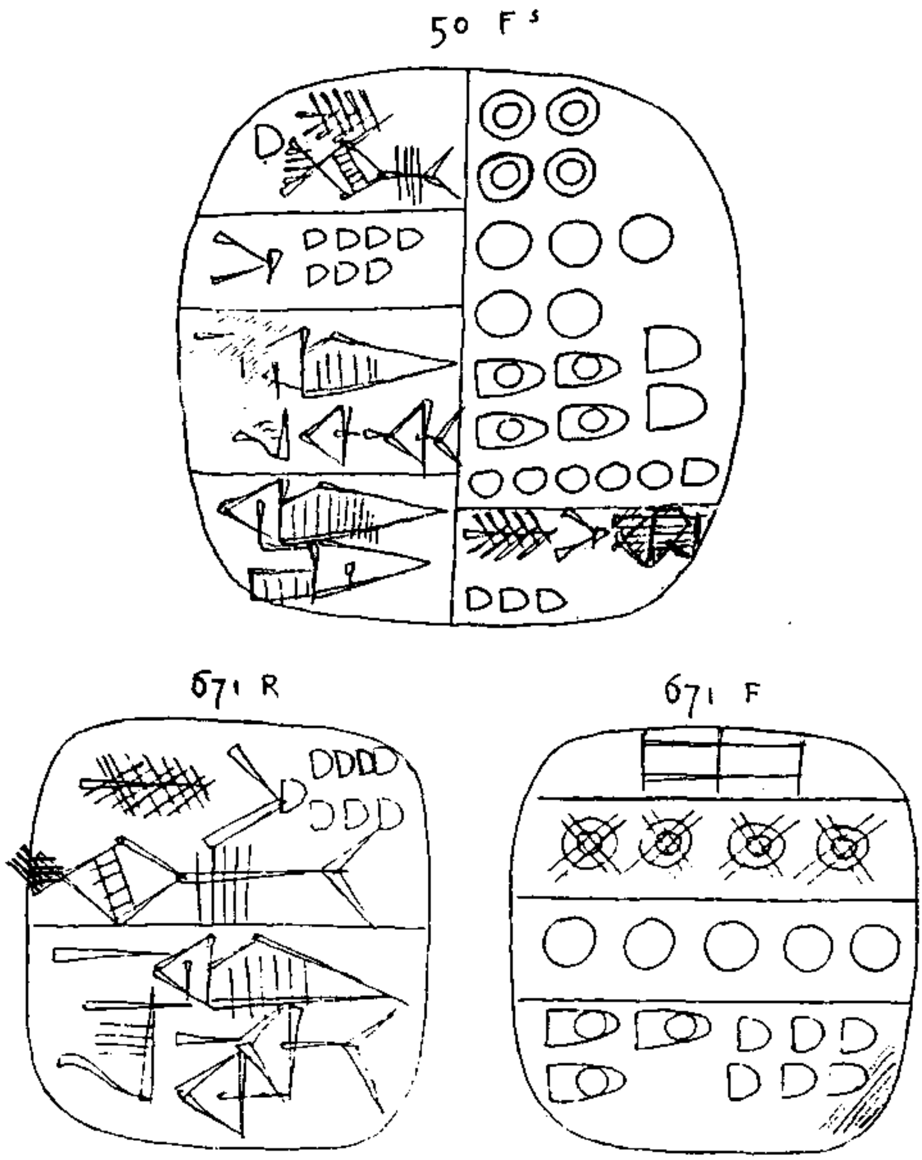

FIG. 1. The two school tablets from c. 2500 B.C. dealing with the distribution of one "granary" of barley in rations of 7 *sila*. The reproductions are basically those of Jestin [1937], but corrections are made where (and only where) Jestin's version was unambiguously contradicted by the photographs (cf. Figs. 2 and 3). Note that No. 671 begins at the reverse of the tablet (if Jestin's identification of obverse and reverse is to be relied upon, as it probably is), and that the result is written in three lines.

with the same problem. The two tablets were originally published by Raymond Jestin [1937, Nos. 50, 671]. They are shown in Fig. 1 in a corrected version of Jestin's somewhat imprecise reproduction, and in photographs in Figs. 2 and 3. In transliterations and translations based on [Powell 1976, 432] the two tablets are as follows [1]:

<table>
<tr><th colspan="2">Jestin 1937, no. 50</th><th colspan="2">Jestin 1937, no. 671</th></tr>
<tr><th>Transliteration</th><th>Translation</th><th>Transliteration</th><th>Translation</th></tr>
<tr><td>še guru$_7$:1</td><td>Grain,
1 granary</td><td rowspan="2">še sila$_3$ 7
guru$_7$</td><td rowspan="2">Grain [in rations of] 7 sila, granary</td></tr>
<tr><td>sila$_3$ 7</td><td>7 sila</td></tr>
<tr><td>lú:1 šu ba-ti</td><td>each man receives</td><td>lú:1 šu ba-ti</td><td>each man receives</td></tr>
<tr><td>lú-bi</td><td>Its men</td><td>guruš</td><td>Workmen</td></tr>
<tr><td>4(šaru)5(šar)
4(g̃ešu)2(g̃eš)
5(u) 1</td><td>45,42,51</td><td>4(šaru)</td><td rowspan="3">45,36,0
[written on three lines]</td></tr>
<tr><td rowspan="2">še sila$_3$:3
šu-tag$_4$</td><td rowspan="2">3 sila of grain left on hand</td><td>5(šar)</td></tr>
<tr><td>3(g̃ešu)6(g̃eš)</td></tr>
</table>

In the above translation, as in the entire text, the Sumerian numerals are translated as normal place value sexagesimals: that is,

$$x,y,z;u,v \text{ means } x \cdot 60^2 + y \cdot 60 + z + u \cdot 60^{-1} + v \cdot 60^{-2}.$$

In this connection one must remember that the place value notation is only attested half a millenium after the Fara texts were written. The numerals in these texts are not written according to a place value notation; instead they make use of special symbols for 1, 10, 60, 600, 3600, and 36000 (see Table 1). Thus, all numbers which are found in the Fara texts (certain simple fractions apart) can be translated into the form X,Y,Z, where X, Y, and Z are integers between 0 and 59 [2]. To remind us of the fact that the place value translation is anachronistic, even if in agreement with established practices, I have used a less orthodox notation in the transliterations.

TABLE 1. The Numerals Used in the Fara Tablets, together with Their Corresponding Number Words.

	= *diš*	= 1	
	= *u*	= 10	
	= *g̃eš*	= 60	= 1,0
	= *g̃ešu*	= 600	= 10,0
	= *šar*	= 3600	= 1,0,0
	= *šaru*	= 36000	= 10,0,0

Note. Observe that the signs as well as the words for 600 and 36000 are composed from 60 · 10 and 3600 · 10, respectively (see [Powell 1972a, 7].)

These numerals are found on the two tablets discussed (see Figs. 1 and 2). The special symbol for 36000 used in No. 671 is probably a mistake; it has slipped in from the system of area notation [Powell 1972b, 218].

TABLE 2. The Basic Capacity Units in Use in the Fara Tablets.

1 sila
1 ban = 10 sila (= 1(u) sila)
1 bariga = 6 ban = 1,0 sila (= 1(g̃eš) sila)
1 gur-maḫ = 8 bariga = 8,0 sila (= 8(g̃eš) sila)

Note. It should be noticed that this system differs somewhat from the systems in use 200 years later [Powell 1976, 423; Lambert 1953, 207].

The *sila* is a measure of capacity (about 1 liter). Tablet 50 is then to be interpreted in the following way: The contents of one storehouse of grain are distributed to a number of men, each man receiving a ration of 7 *sila*. Consequently, 45,42,51 rations (i.e., 164571 rations) are distributed, and a remainder of 3 *sila* is left over. As pointed out by Lambert ([1953, 206] and a correction in [1954, 150]), this presupposes that the contents of a standard granary (or some specific granary which is dealt with in the exercise [3]) is known. Further, if the exercise is performed correctly we may conclude that the contents of the granary should be 40,0 *gur* (4 (*g̃ešu*) *gur*), with the *gur* in question (the *gur-maḫ*) equal to 8,0 *sila* in Fara (see Table 2 and [Lambert 1953, 206 f.]).

Tablet 671 would of course lead to a different result. However, this tablet has obviously been written by a much less competent student, and since, furthermore, the resulting magnitude of the contents of the granary is much less simple, we may confidently follow tablet 50.

The first implication of this concerns metrology. The very simple structure of the granary led Lambert [1953, 207] to include it in the list of metrological units used in Fara. According to A. A. Vaiman, however, this is very doubtful since such a use of the term is attested nowhere else [Marvin A. Powell, private communication]. On the other hand, the use of the term *guru*$_7$ used in two parallel, somewhat later royal inscriptions from Lagaš to designate not the building but the quantity of grain which can fill the granary, seems beyond doubt. (Entemena, Cone A, II^{21}, II^{25}, IV^{11}, and Cone B, corresponding places; [Sollberger 1956, 37 f.; and Thureau-Dangin 1907, 38-41]). Upon this point there is agreement between the otherwise wildly divergent interpretations of the passages in question [4]. Toward the end of the third millennium, the term meant 1 (*šar*) royal *gur* of 300 *sila* each [Thureau-Dangin 1932, 39], that is, 1080000 *sila*, which is quite close to but yet different from our assumed Fara value. Thus, it might appear that the best conjecture is that the *guru*$_7$ was not a real metrological unit, but rather a standard expectation concerning the contents of a physical storehouse, relatively unaffected by changing metrological conventions but always expressed in round numbers in the current metrology.

In any case, it must be regarded as justified to interpret the *guru*$_7$ in *our* tablets as 4 (*g̃ešu*) *gur*, or, equivalently, as 5,20,0,0 (=1152000) *sila* (in the rest of the paper I shall assume this result). The problem dealt with on the tablets is therefore a *formal division problem*. It is probably the oldest such problem known, even though practical division problems presumably must have presented themselves at a much earlier date to the temple administrations. So, the method used to solve the problem is of some interest in connection with the problem of the development of Mesopotamian mathematics--not least because later Mesopotamian mathematics differs from all other mathematical traditions by its use of reciprocals for divisions.

FIG. 2. Tablet 50. At bottom the lower right corner of the tablet ("3 *sila* of grain left on hand"). Photographs kindly supplied by Istanbul Arkeoloji Müzeleri Müdürlüğü.

Tablet 50 (Fig. 2) was first discussed at length by Geneviève Guitel [1963]. She proposed--mainly because this is a way in which the division *can* be performed and because of the visual impression offered by the arrangement of the result--that the scribe "ait utilisé une méthode absolument analogue à la pratique moderne d'une telle division," calculating the number of *sila* to a *guru*$_7$ and expressing it as 32 *šaru* [5], and then performing a normal "long division" through the levels defined by the successive numeral symbols. As long as only a statement of the problem and a correct solution were known, nothing better could be done.

Powell [1976, 432 f.] discovered that tablet 671 contains the same problem, assuming a copying error on the part of Jestin. (Such an error is confirmed by the photograph.) This parallel, but erroneous, calculation was used by Powell as the basis of a new approach. He found Guitel's absolutely modern method suspicious, proposing instead the possibility that the Fara scribes used the only method of division attested in later Babylonian mathematics, namely, multiplication by the reciprocal. Powell correctly pointed out that in order to obtain the correct answer (that of No. 50), 1/7 has to be calculated to four sexagesimal places. Since 1 *guru*$_7$ = 5,20,0,0 *sila*, the procedure would be as follows:

(1) $5,20,0,0 \cdot 0;8,34,17,8 = 45,42,51;22,40,$

(2) $45,42,51 \cdot 7 = 5,19,59,57,$

(3) $5,20,0,0 - 5,19,59,57 = 3.$

So, the number of men is 45,42,51, and 3 *sila* are left over--just as stated in No. 50.

This calculation was carried out in the full sexagesimal system where it can, of course, be made. For the moment we leave aside the question whether such a calculation could be made by the Fara scribes.

Apart from historical continuity the main support for Powell's proposal is that it makes sense of the wrong result of No. 671: As stated by Powell, this wrong result is obtained if one uses 0;8,33 (=57/400) instead of 0;8,34,17,8, ... for 1/7; then the number of men is found to be 45,36,0, and no remainder is left. If no other explanation of this result could be found, this would support Powell's interpretation, and consequently the idea that an equivalent of the full sexagesimal system was in use in Fara. In any case, after Powell's parallelization of the two texts it should no longer be possible to neglect the error in No. 671 as a source of information [6].

Reverse

Obverse

Obverse, lower edge

Reverse, upper edge

FIG. 3. Tablet 671. Photographs kindly supplied by Istanbul Arkeoloji Müzeleri Müdürlüğü.

There is, however, another way to carry out the calculation which not only makes the error committed on tablet 671 possible (as does the use of reciprocals), but also implies that this error is one of the most obvious of all possible errors.

This other method is suggested by the fact that quantities of grain greater than the *gur* were measured in *gur* and expressed by the standard numerals listed in Table 1, while smaller quantities were measured by subunits designated by a special series of symbols, a series which runs parallel to the series of standard numerals but which does not contain the *gur* and its multiples as round sexagesimal multiples of the smallest unit, the *sila* (see Table 2). We may therefore suppose that the first way in which the student would think of the granary would be as 4 (*g̃ešu*) *gur*. If we divide this number by 7, the result is 5,42 and a remainder of 6 *gur*. If we forget about the remainder and multiply 5,42 by the number of *sila* to a *gur*, we get 45,36,0--just the result of No. 671!

Few other procedures lead to this result in a correspondingly simple way. Those I have been able to devise seem too artificial to be taken seriously into account [7]. *Thus, a plausible explanation of No. 671 is that the scribe divided the number of* gur *by 7, forgot about the remainder, and multiplied by 8,0.*

Truly, No. 671 was written by "a bungler who did not know the front from the back of his tablet, did not know the difference between standard numerical notation and area notation, and succeeded in making half a dozen writing errors in as many lines" [Powell 1976, 432]. However, several indirect arguments suggest that the able student who wrote No. 50 followed the same path as the bungler, though with greater success.

First, from all we know about the methods of Mesopotamian mathematics education, it consisted largely of working out specific problems, most often with many examples of similar type. It is not likely that the methods of the Fara school were at a theoretical level higher than those of the Old Babylonian scribal school--in other respects, at least, Old Babylonian teaching followed the very characteristic pattern which had been created many centuries before the Fara tablets were written (see [Falkenstein 1936, 46 f.]). Of course, teaching by means of examples does not prevent students from understanding the mathematical principles involved; but in most cases such an understanding will not inspire the average student (and a fortiori not the dunce or the beginner) to invent a method radically different from the standard one. A student might err by dividing the number of *gur* instead of the number of *sila*, but then it is unlikely that he would almost rectify his error by multiplying by the number of *sila* to a *gur*.

Second, the above-mentioned custom of expressing large volumes in *gur* rather than in *sila* coincides with what appears to be the general method of thinking of large quantities. Certainly, the same applies to area measures: "The notation beginning with *bur* and its higher multiples runs exactly parallel to standard numerical notation" [Powell 1972b, 175]. This general tendency--to count in terms of the greatest unit--makes a calculation in two steps much more natural than a conversion of the $guru_7$ directly to *sila*, provided, of course, that the Sumerians could conceive of and perform the calculation correctly in two steps. (This is discussed below.)

Third, we may ponder the possible alternative strategies. Conversion to *sila* followed by a "long division," discussed above and in [6], appears to be incompatible with the result of No. 671. Another alternative is the one proposed by Powell.

It will appear from the discussion of No. 50 (above) that this calculation cannot be performed by the numerals which have come down to us from the Fara period (Table 1); even circumvention strategies such as the factorizations suggested by Guitel and Bruins will not do. This, however, is an objection of restricted value: For one thing, the numerals found on the material which has come down to us may very well be incomplete. While small numbers occur on most tablets from Fara, the *šar* and the *šaru* occur so rarely that it may well be that still higher numerals either have not survived or have escaped notice completely [8].

It is less probable, although not impossible, that an alternative notation system, with genuine place value features, was already in existence; in fact, even in Ur III where the place value notation is attested this system is an alternative notation, used for marginal and intermediate calculations (see [Ellis 1970, 267 f.; Powell 1976, 420 f., 435 n. 6]). But the existence of a real place value system is not a necessary precondition for the performance of division via multiplication by the reciprocal. Even in the Fara notation, as we know it, simple divisions may well have been performed by means of "multiplicative complements" followed by a shift of sexagesimal "order of magnitude" [9].

Finally, we should remember that numbers can, for the purpose of calculation, be represented by means other than written notations. Denise Schmandt-Besserat's recent discoveries [1977, 1978, 1979] of the pre-Sumerian use of small clay tokens to represent numbers, points to a possible basis for an abacus-like representation (requiring not necessarily an abacus board or frame). Moreover, archives from Nuzi from the second millenium B.C. show that material counters, probably related to the pre-Sumerian ones, were used for recording purposes even at this time [Oppenheim 1959] [10]. On the other hand, in a problem discussed by Powell [1976, 426 f.] a student from c. 2200 B.C. is apparently groping after the basic idea of the extension as libitum of sexagesimal notation; the way in which he loses track of the correct sexagesimal place suggests that he was working with a mental construct and not with a notation or a material representation.

All in all, we must conclude that the existence in the Fara period of techniques permitting a solution of our problem in agreement with Powell's suggestion remains an unproven hypothesis. Therefore, the assumption that the scribes responsible for the two tablets followed the same path--that of a division in two steps--seems all the more likely.

However, before this assumption is accepted two questions must be asked: Did the Fara scribes possess the technical competence to carry out the correct solution given on tablet 50 by a division in two steps? And, assuming this technical competence, is it reasonable to assume that they would have thought of using a division in two steps?

The result obtained in No. 671 will help us answer the first question: Assuming that the two-step interpretation of this wrong result is correct, *40,0 is correctly divided by 7.* (Although the remainder is dropped, we may assume that the remainder could be found by any competent scribe--if it was not yielded directly by the calculation, it could be found by multiplication and subtraction once the quotient was known.) Also, *5,42 is correctly multiplied by 8,0.* the result being 45,36,0.

Would these skills suffice technically to solve the problem entirely? Probably in the following way: 40,0 *gur* are divided by 7. The result is 5,42, with a remainder of 6 *gur*. The 5,42 are multiplied by 8,0 (the number of *sila* to a *gur*), and the result is 45,36,0 (men). So far we have done precisely what seems to have been done on No. 671. The remainder is converted to 6·8,0 = 48,0 *sila*; such a step does not explicitly appear in No. 671, but the ability to perform it is inherent in the metrological systems used by the Fara scribes in their accounting. What sense would be left to such systems if a unit could not be converted to a smaller one? Further, the conversion is conceptually related to the multiplication by 8,0 in No. 671, even if this is not a simple conversion of units.

The remaining 48,0 *sila* are then divided by 7; this step is similar to the division of 40,0 by 7 in No. 671. The result is 6,51 (men), which is added to the 45,36,0 (men) already found; this addition is not different from the addition probably inherent in the multiplication of 5,42 by 8,0, and neither does it differ from the bulk of additions necessary in ordinary contemporary accounting. The result is 45,42,51 men, as stated in No. 50. Besides, 3 *sila* are left over, as also stated. So, the method and the abilities revealed in No. 671 indicate sufficient *technical* ability to produce the correct result of No. 50. Moreover, this analysis implies that the *apparent* result of No. 671 is not necessarily a wrong *final result*; it may be an intermediate result in a calculation which the student did not know how to complete, or which he was unable to complete on this tablet where no empty space was left (see Fig. 1 and 3).

Technical ability is one thing, how to use it another. Would the Sumerians have been able to *conceive the idea* of a division in two steps? Does this not require the explicit recognition that multiplication and division are interchangeable mathematical procedures? And is such a recognition to be expected at this early stage?

Of course we cannot exclude the possibility of this recognition. But a comparison with Middle Kingdom Egyptian mathematics suggests that it is not to be expected: Middle Kingdom mathematics constitutes a coherent and relatively well-developed structure [11]. Still, the vocabulary which it used until Demotic times to describe multiplication and division is fetched from counting [Peet 1923, 22 f.; Parker 1972, 7 f.]; the Egyptians seem not to have thought of these as independent procedures.

On the other hand, the Sumerians need not have thought of interchangeable procedures in order to have conceived the idea of a two-step division. The method can just as well be described quite intuitively, as follows: One man gets 7 *sila*. Since 1 *gur* is 8,0 *sila*, then 8,0 men will receive 7 *gur*. 7 *gur* are contained 5,42 times in a *guru*$_7$ of 40,0 *gur*, and 6 *gur* are left over. So, 8,0 men can be paid 5,42 times, and the 6 *gur* may provide for a supplementary number of men, etc. [12].

This pattern of thought is so close to concrete experience that it should be accessible to a mathematical culture in which problems such as No. 50 could be solved. However, it does not follow from the simplicity of the argument that precisely this form of reasoning was used, but only that a simple argument *could* produce the two-step calculation. In any case, we may say that *the two-step-calculation is a well-supported hypothesis*.

CONCLUSION

How much does this analysis of the tablets reveal about the general character of mathematical thought of the Fara period? More specifically, will it tell whether the concept of place value was already on the way? And further, can we be confident that the methods of the specific problems found on the two tablets reflect the normal customs of the time?

No traces of a concept of place value were revealed in the treatment of the problem. So, the tablets do not indicate that place value was already on its way. This absence of positive evidence is, however, inconclusive, because the problem dealt with on the tablets is peculiar in several respects:

> *The quantity to be divided is very large if expressed in* sila.
>
> *On the other hand, the* gur *probably presented itself as a* natural *intermediate step; it may be that the* guru$_7$ *was spontaneously thought* of only *as consisting of a number of* gur*--just as we think of a foot only as consisting of 12 inches, not of 144 lines.*
>
> *The divisor 7 is* irregular; *i.e., it does not divide 60 or any power of 60. As pointed out by Powell* [*1976, 433*]*, 7 may very well have been chosen for the exercise* because *it was irregular.*

So, although analysis of the tablets suggests a mathematical mode of thought closely connected to current metrology and not yet familiar with the concept of place value, it does not necessarily imply that these were general characteristics of Fara mathematics. Metrology *may* simply have been used to circumvent the particular difficulties of this specific problem. Other calculations--in particular, the intermediate calculations used in the solution of the problem (40,0 ÷ 7, 5,42 · 40,0, etc.)--may, but need not, have been performed metrologically; may, but need not, have been performed by means of nonwritten representations of numbers [13]; and may, but need not, have been performed by place value-related reasoning. The wide variety of elementary mathematical techniques--many different from ours--which are known from other places and epochs [14] demonstrates that there are numerous possible ways to solve the same numerical problem.

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NOTES

1. The reading *šu-tag$_4$* proposed as a conjecture by Powell on the basis of Jestin's reproduction should according to Aage Westenholz (private communication) be consolidated by the photographs. According to later work by Powell [1978, 182 ff.], the reading *tag$_4$* should presumably be replaced by the reading *taka*. The interpretation "left on hand" is supported by this analysis.

2. Here I have omitted all considerations of the various quasi numbers used in various metrological systems (e.g., [Powell 1971; 1972b; Friberg 1978]; most of them are structurally similar to the quasi-sexagesimal system shown in Table 1.

3. Many things, not least of which is the enormous number of rations, point to the tablet's being in fact a mathematical exercise and not an administrative calculation.

4. Thus, according to Thureau-Dangin [1907, 39] and Sollberger and Kupper [1971, 72], Cone A, IV, 11, means "1 *šar guru*$_7$s"; according to Barton [1929, 63] and Lambert [1956, 143 n. 1], the meaning is "1 *šar* [*gurs*, i.e., one] *guru*$_7$." The difference amounts to that between realistic and imaginary quantities of grain. Still, everyone agrees that definite quantities of grain are meant.

5. Such factorizations have often been used in other historical contexts when there was the problem of a restricted numeral system, e.g., Ancient Egypt [Sethe 1916, 9], Shang China [Needham 1959, 13], Ancient Rome [Friedlein 1868], and the early Latin abacists as well as vernacular mathematical treatises of Medieval Western Europe [Bubnov 1899, 203-209 and *passim*; Henry 1882, 67 f.]. So, even the Sumerians may very well have hit upon the trick--indeed, a similar way of looking at things is suggested by the multiplicative structure of the number words *g̃eš-u* and *šar-u* and the corresponding written symbols (see Table 1).

6. Yet this was done by Bruins [1978] in a critical abstract of Powell's paper. In principle, Bruins went back to Guitel's interpretation, with the difference that either he factorized 5,20,0,0 as 5,20 *šar*, or assumed the use of the full sexagesimal system for integers. Bruins' argument for this use of long division was based partly on the writing of the result of No. 671 in three lines. He neglected, however, to note that this writing is unsystematic; *šaru* and *šar* are separated, while *g̃ešu* and *g̃eš* are written together. He also assumed (advancing it as a fact) that for "the division by irregular numbers a table of multiples is made"--apparently an extrapolation from the much more recent Old Babylonian multiplication tables.

7. One is the deliberate use of 57/400 as an approximation to 1/7. Since such an approximation is clearly not used in No. 50, I would discard it. The other possibilities all consist in the measurement of the *guru*$_7$ in terms of units other than the *gur-maḫ*: 576 *sila*, 720 *sila*, 960 *sila*, 1440 *sila*, or 2880 *sila*. None of these occurs in the rather meager evidence for Fara metrology, although all are simple multiples of the *gur-maḫ*, of the *gur* of 240 *sila* (also found in Fara), or of the *gur* of 144 *sila* known from later Lagaš (see [Powell 1976, 423; Lambert 1953, 205 f.]). Measurement in terms of any one of these magnitudes is not likely to have occurred, *unless* that magnitude was (unknown to us) a metrological unit. If this were the case,

however, we would, in principle, be brought back to the division in two steps, but using another *gur* rather than the *gur-maḫ*.

8. A preliminary and elementary statistical analysis suggests that the existence of unnoticed higher numerals is improbable, but far from impossible. Of the first 173 tablets in Jestin's randomly organized collection [1937], about 162 contain numerals or metrological symbols (the precision is limited because of Jestin's not always quite reliable reproductions). "1" is present on about 145 tablets, "10" on about 75, "60" on about 27, "600" on about 21, "3600" on about 10, and "36000" on about 3. This fits beautifully to a straight line in a log-log diagram. By extrapolation, the next numeral of the series ("216000") should be expected to occur on approximately 1/2 to 1% of all tablets, if it existed. On the other hand, the only argument for the use of log-log analysis is that it works between "1" and "36000".

9. It is possible, perhaps even plausible, that one of the motives for the introduction and general adoption of the full place value system was the fact that it permitted the generalization of the principle of multiplicative complementarity. (I have discussed this in another connection in [Høyrup 1980, 19; 84 ff., nn. 37, 38, 43].)

10. Surely, one should not conclude too much from the Nuzi find. Nuzi was a Hurrian city and need not have inherited its seemingly rather primitive administrative techniques through Sumer. True enough, mid-third-millenium clay tokens are found in Ur, Kish, and a number of other cities ([Schmandt-Besserat 1977, 9 f., 14, 20]; and Schmandt-Besserat, private communication). However, using the published background references for Ur and Kish [Woolley 1934; MacKay 1929], I have found nothing connecting these finds with any specific use, except, on one hand, an interesting similarity between the tokens and the men and dice used in board games (cf. [Woolley 1934, I, 175-178; II, pl. 95, 98, 158] and, on the other hand, the fact that all third-millenium tokens seem to belong to categories with a numerical, rather than a conceptual, significance. In any case, the very rich token system of the fourth millenium again becomes very simple (Schmandt-Besserat, private communication). In a recent publication, Stephen J. Lieberman [1980] suggests that the tokens were used as calculi in an abacus without a counting board. Lieberman introduces some interesting considerations involving the use of the two different ways to write numbers ("curviform" and "cuneiform": see the drawings in [Powell 1972a]) in later Sumerian accounting practice but he does not decide (nor even mention that it is a problem) whether the hypothetical abacus was analogous to the later place value system (this is almost claimed on p. 342), or it was isomorphic with the system of curviform numerals as shown in Table 1 (this isomorphism is implied by the arguments on pp. 344 f.).

11. The cumbersome character of the Egyptian unit fraction calculations should not be taken as evidence for lack of mathematical coherence. Further discussion of the character of Egyptian mathematics may be found in [Høyrup 1980, 31 ff.].

12. I am told by Marvin Powell that A. Vaiman of the Leningrad Hermitage has mentioned a similar method of thinking to him (private communication).

13. One possibility is the use of some abacus-like representation, for example, one based on Schmandt-Besserat's clay tokens. Another is the *finger reckoning* of the Ancient and Muslim world, which the Muslims thought of as "arithmetic of the Byzantines and the Arabs" [Saidan 1974, 367], and which Saidan conjectures to have descended from Greco-Babylonian manipulational practices. (Still, an Egyptian cubit rod, reproduced by Karl Menninger [1958, 23], carrying pictures of finger positions instead of the corresponding numbers suggests that Egypt may be a more plausible origin for Greek and Muslim finger reckoning than Mesopotamia.)

14. A few examples should be mentioned, all belonging to unsophisticated mathematical cultures:

The Ancient Egyptian multiplication by duplations and division by filling-out;

the "Russian peasant multiplication" described by Plakhovo [1897], which is related to, but yet different from, the Egyption method;

the awkward but very down-to-earth division practiced on the Medieval "Gerbert" abacus (see, for instance, [Smith 1925, 134 f.] or [Yeldham 1926, 42 ff.]).

REFERENCES

Barton, G. A. 1929. *The royal inscriptions of Sumer and Akkad*. New Haven, Conn.: Yale Univ. Press for the American Oriental Society.

Bruins, E. M. 1978. (Critical abstract of [Powell 1976].) *Zentralblatt für Mathematik und ihre Grenzgebiete* 357, 7-8.

Bubnov, N. (ed.). 1899. *Gerberti postea Silvestri II papae Opera Mathematica (972-1003)*. Berlin: Friedländer. Reprint, Hildesheim: Georg Olm, 1963.

Ellis, M. de J. 1970. A note on the "Chariots crescent." *Journal of the American Oriental Society* 90, 266-269.

Falkenstein, A. 1936. *Archaische Texte aus Uruk* (Ausgrabungen der Deutschen Forschungsgemeinschaft in Uruk-Warka, Bd. 2). Berlin: Deutsche Forschungsgemeinschaft.

Friberg, J. 1978. The third millenium roots of Babylonian mathematics. I. A method for the decipherment, through mathematical and metrological analysis, of proto-Sumerian and proto-Elamite semi-pictographic inscriptions. Preprint, Department of Mathematics, Chalmers University of Technology and the University of Göteborg.

Friedlein, G. 1868. De notis numerorum romanis. *Bulletino di Bibliografia e di Storia delle Scienze mathematiche e fisiche* 1, 48-51.

Guitel, G. 1963. Signification mathématique d'une tablette sumérienne. *Revue d'Assyriologie* 57, 145-150.

Henry, C. 1882. Sur les deux plus anciens traités français d'algorisme et de géométrie. *Bulletino di Bibliografia e di Storia delle Scienze matematiche e fisiche* 15, 49-70.

Høyrup, J. 1980. Influences of institutionalized mathematics teaching on the development and organization of mathematical thought in the pre-modern period. *Materialien und Studien. Institut für Didaktik der Mathematik der Universität Bielefeld* 20, 7-137.

Jestin, R. R. 1937. *Tablettes sumériennes de Šuruppak au Musée de Stamboul* (Mémoires de l'Institut Français d'Archéologie de Stamboul, III). Paris: Boccard.

Lambert, M. 1953. La vie économique à Shuruppak. *Sumer* 9, 198-213.

——— 1954. La vie économique à Shuruppak (II). *Sumer* 10, 150-190.

——— 1956. Une histoire du conflit entre Lagash et Umma. *Revue d'Assyriologie et d'Archéologie Orientale* 50, 141-146.

Lieberman, S. J. 1980. Of clay pebbles, hollow clay balls, and writing: A Sumerian view. *American Journal of Archaeology* 84, 339-358.

Mackay, E. 1929. *A Sumerian palace and the "A"-Cemetery at Kish, Mesopotamia,* part II (Field Museum of Natural History, Anthropology, Memoirs, Vol. I, No. 2). Chicago.

Menninger, K. 1958. *Zahlwort und Ziffer. Eine Kulturgeschichte der Zahl.* II. *Zahlschrift und Rechnen.* Göttingen: Vandenhoech & Ruprecht.

Needham, J. 1959. *Science and civilisation in China,* Vol. 3, *Mathematics and the sciences of the heavens and the earth.* Cambridge: Cambridge Univ. Press.

Oppenheim, A. L. 1959. On an operational device in Mesopotamian bureaucracy. *Journal of Near Eastern Studies* 18, 121-128.

Parker, R. A. 1972. *Demotic mathematical papyri.* Providence, R.I./London: Brown Univ. Press.

Peet, T. E. 1923. *The Rhind mathematical papyrus, British Museum 10057 and 10058. Introduction, transcription, translation and commentary.* London: Univ. Press of Liverpool.

Plakhovo. 1897. [Report on a note by Plakhovo in JME (Journal de Mathématique Élémentaire?) on "la multiplication égyptienne et russe"]. *Mathesis* (Gand) 17, 86-87.

Powell, M. A. 1971. *Sumerian numeration and metrology.* Dissertation, The University of Minnesota. Reprint, Ann Arbor, Mich.: Univ. Microfilms International, 1973.

——— 1972a. The origin of the sexagesimal system: The interaction of language and writing. *Visible Language* 6, 5-18.

——— 1972b. Sumerian area measures and the alleged decimal substratum. *Zeitschrift für Assyriologie und vorderasiatische Archäologie* 62, 165-221.

——— 1976. The antecedents of Old Babylonian place notation and the early history of Babylonian mathematics. *Historia Mathematica* 3, 417-439.

——— 1978. Ubuki to mother ... The situation is desparate: A plaidoyer for methodological rigor in editing and interpreting Sumerian texts with an excursus on the verb taka: da_x - da_x (TAG_4). *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* 68, 163-195.

Saidan, A. S. 1974. The arithmetic of Abū'l-Wafā'. *Isis* 65, 367-375.

Schmandt-Besserat, D. 1977. An archaic recording system and the origin of writing. *Syro-Mesopotamian Studies* 1, No. 2.

——— 1978. The earliest precursor of writing. *Scientific American* 238, No. 6, 38-47.

——— 1979. An archaic recording system in the Uruk-Jemdet Nasr period. *American Journal of Archaeology* 83, 19-48.

Sethe, K. 1916. *Von Zahlen und Zahlworten bei den Alten Ägyptern, und was für andere Völker und Sprachen daraus zu lernen ist* (Schriften der Wissenschaftlichen Gesellschaft Strassburg, 25. Heft). Strassburg: Trübner.

Smith, D. E. 1925. *History of Mathematics. II. Special topics of elementary mathematics.* Boston: Ginn. Reprint of last edition, New York: Dover, 1958 (pagination refers to this reprint).

Sollberger, E. 1956. *Corpus des inscriptions "royales" présargoniques de Lagaš.* Genève: Droz.

——— & Kupper, J.-R. 1971. *Inscriptions royales sumériennes et akkadiennes.* Paris: Les Éditions du Cerf.

Thureau-Dangin, F. 1907. *Die sumerischen und akkadischen Königsinschriften* (Vorderasiatische Bibliothek, I. Band, Abteilung 1). Leipzig: J. C. Hinrichs'sche Buchhandlung.

——— 1932. *Esquisse d'une histoire du système sexagésimal.* Paris: Geuthner.

Woolley, C. L. 1934. *Ur excavations,* Vol. II, *The Royal Cemetery.* Part I: *Text;* Part II: *Plates* (Publications of the Joint Expedition of the British Museum and the Museum of the University of Pennsylvania to Mesopotamia). London.

Yeldham, F. 1926. *The story of reckoning in the Middle Ages.* London: Harrap.