On Old Babylonian mathematical terminology and its transformations in the mathematics of later periods

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"... wo Begriffe fehlen, da stellt ein Wort zur rechten Zeit sich ein", thus Mephisto in Goethe's *Faust* (I, 1995*f*). This may be true in (pseudo-)sciences like theology (of which Mephisto speaks) and philosophy, and according to certain philosophies of mathematics, considering only the formal game of symbols signifying nothing beyond their appearance within axioms, about this "queen and handmaid of science". However, exactly this epithet raises Eugene Wigner's famous question [1960] about "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" – in order to be effective (not to speak of *unreasonably* effective), the words / terms of mathematics need correspond to concepts, not only understood as a network of operations within a space defined by outspoken or tacitly assumed axioms but also as networks that reach beyond the space of abstract beer-mugs, chairs and tables attributed to Hilbert. When they do not, we get instead "unreasonable *in*effectiveness", as K. Vela Velupillai [2005: 849] states about mathematics in economics: "Unreasonable, because the mathematical assumptions are economically unwarranted; ineffective because the mathematical formalisations imply non-constructive and uncomputable structures".

Certainly, when it comes to Mesopotamian mathematics we know the concepts and the operations almost exclusively through the words of texts – the exceptions being some geometrical drawings; some weights and measuring sticks corresponding to metrological units; some tables of technical constants that must be understood within the limits of the physically possible or in agreement with artefacts (bricks etc.) that have been excavated; in Late Babylonian times (in mathematical astronomy) in agreement with celestial phenomena which we know in other ways; and a bit more. Our own knowledge about the structure of elementary arithmetic and elementary Euclidean geometry may also help us (tables of reciprocals stating that the i g i does not exist or simply omitting this line correspond well to our idea that 7 does not divide any power of 60), but should of course be used with care.

None the less, Mephisto and the crash between Wigner and the folklore Hilbert should warn us that remaining within the walled magic garden of words may delude.

Long-living Practices

So, let us start with two long-living *practices*, reflected in words that change. The first has to do with the determination of the circular perimeter from the diameter. In Mesopotamia, the ratio between the two magnitudes was supposed

to be 3:1.[1] Mostly, the basic circle perimeter was the circumference, but in cases where it has to be found from the diameter[2], the operation is not a "raising" (*našûm* / í l) as one would expect from this being the operation invariably used in multiplication with technical constants (see below, p. 24) but a *tripling* (*šullušum*) – or the diameter is "repeated in three steps".

This could be an unimportant though unexplainable quirk, but Greek practical geometry as contained in the pseudo-Heronian *Geometrica*[3] and as quoted by Heron in the *Metrica*[4] shows that it is not. On all occasions, the terms τρισσάκις and τριπλάσιον are used even when neighbouring multiplications are ἐπὶ *n*; afterwards, a supplementary Archimedean seventh is added.[5] Even *if* we believe that the Greek practitioners had translated the verbal rules of Mesopotamian forerunners, this level of philological precision would be astonishing.

The explanation is found in two texts from the late Middle Age, one in Middle High German and one in Old Icelandic. The former is Mathes Roriczer's

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[1] Old Babylonian scribes were probably aware that this was a practical value or approximation, since they also knew 3:1 to be the ratio between the perimeter and the diameter of a regular hexagon [TMS, 24]. Alternatively, the value has recently been proposed [Brunke 2011: 113] "to be the result of a specific Babylonian way to *define* the area measure of a circle". Since nothing suggests the Babylonians to have bothered about mathematical definitions, this idea can probably be discarded.

Whether the possible alternative ratio $3\frac{1}{8}$:1 suggested by the text YBC 8600 [MCT, 57–59] was supposed to be a better approximation or was just adopted (*if* it was really meant) for ease of calculation (as supposed by Otto Neugebauer and Abraham Sachs) is hardly decidable. The suggestion of E. M. Bruins to find the same approximation in an i g i . g u b table (a table of technical constants) from Susa [TMS, 26, 28] can be discarded, since š á r means neither "circle" not "more perfect circle".

[2] BM 85194, obv. I 47-48; Haddad 104, I 4, 14, 26, 40, II 7, 26, 36, 42, III 14, 20, 26.

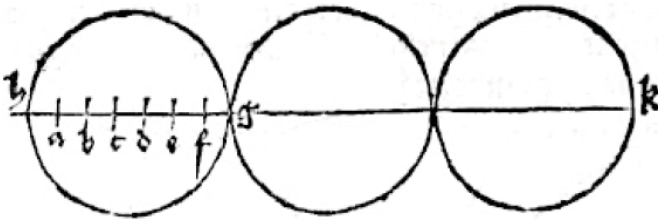
[3] These treatises were published by Heiberg [1912] as one, even though he clearly saw and expressed [1914: xxi] that at least two independent treatises are involved; I suppose that the structure of the *Opera omnia* was already determined when he was called into the project at Wilhelm Schmidt's death, even though this is not said clearly in the beginning of the introduction [Heiberg 1912: iii*f*]. In any case, the bulk of the conglomerate comes from two treatises (even they composite) represented most fully by Mss AC and MS S. The relevant passages are Mss AC, 17.10, 17.29; SV: 17.8 S:22.16; and S:24.45 (S.24 is actually a third small but still composite treatise). See [Høyrup 1997].

[4] I.xxx, xxxi, ed. [Schöne 1903: $74^{5,25}$].

[5] Except in the *Metrica*, where Heron distinguishes "the ancients" who took the perimeter to be the triple of the diameter, and the recent workers who take it to be the triple, and one seventh added.

Geometria deutsch from c. 1486 or shortly afterwards. This is how it tells how to make a round line straight [Roriczer 1497: 4[r]]:[6]

> Nernach so einer ein gerunden riss scheitrecht machen wil dz d scheitgerecht ris und dz gerund ain leng sey. So mach drey gerunde neben ein ander und tayl dz erst rund in siben gleiche teil mit den puchstaben verzeichnet h.a.b.c.d.e.f.g: Darnach alsz weit vom .h. in das .a. ist da setz hindersich ein punckt da setz ein .i. Darnach als weit von dem .i. pisz zu dem .k. ist Gleich so lang ist der runden riss einer in seiner rundung der drey neben ein and sten des ein figur hernach gemacht stet.



In literal translation:

> When somebody wishes to make a round line straight, so that the straight line and the round are one length. Then make three rounds next to one another, and divide the first round into seven equal parts, designated with the letters *h a b c d e f g*. Then as far as it is from *h* to *a*, set behind it a point, and set an *i*. Then as far as it is from the *i* to the *k*, so long is one of the rounds in its rounding of the three that stand next to each other, of which a figure stands made hereafter.

The old Icelandic manuscript A.M. 415 4to from the early fourteenth century, on its part, states (fol. 9[v]) that "the measure around the circle is three times as long as its width, and a seventh of the fourth width",[7] obviously a reference to a similar construction.

As we see, the medieval texts tell how to *construct* the length of the perimeter, not how to calculate it. This construction must have been used by master builders at least from Old Babylonian times until late medieval gothics, with only a marginal change taking into account Archimedes's improved approximations. A practical construction, not philological precision, explains the accuracy of the "translation" of the rule.

The other example remains within the Mesopotamian orbit. As shown by Christine Proust [2000], Mesopotamian calculators made use of a reckoning board called "the hand", from the time of Shuruppak until that of the Seleucid astronomers. The name (š u / *qātum*) is likely to have been transmitted at the level

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[6] According to [Shelby 1977: 120*f*], this differs from Roriczer's original only in orthography.

[7] "Ummæling hrings hvers þrimr lutum lengri en bréidd hans ok sjaundungr of enni fiorðo breidd" [Beckman & Kålund 1914: 231*f*].

of words (unless we imagine that a real hand can have been used to carry five levels for ones and five levels for tens and permit easy transfer of "calculi" between these ten handheld cases).

However, a strange continuity at the level of semantics seems to be better explained at the level of operations. As we shall see, Old Babylonian texts use ĝ a r . ĝ a r and z i logographically for *kamārum* and *nasāḫum*, respectively, that is, for "heaping" addition and subtraction by removal (cf. below). However, a well-known passage from "Šulgi-Hymn B", l.17 [ed. Castellino 1972: 32] claims that the king has learned z i . z i g á . g á š i d n ì . š i d , "to subtract and add, counting and accounting".[8] z i . z i and ĝ á . ĝ á are *marû*-stems of z ì g , "to rise", and ĝ a r , "to place", respectively [Thomsen 1984: 305, 322], and probably mean "to take up" and "to put down" – namely on the reckoning board. These are not the meanings of *nasāḫum* and *kamārum*, and it appears that the Sumerograms have been selected for semantic proximity, not identity (as happened in other cases, too), and even abbreviated (into z i) or changed (into ĝ a r . ĝ a r).

In a small batch of mathematical texts produced in the environment of scholar-scribes in the fifth century[9] (see below, p. 31), subtraction is spoken of as n i m , which in Old Babylonian texts occurs occasionally as a logogram for the "raising" multiplication (belonging to the same semantic cluster as *našûm* and í l , see below, p. 24). In the fifth century, the meaning seems to be "to take up" or "lift", that is, to refer once again to the reckoning board.

In the Seleucid text BM 34568, we similarly find for instance "16 t a 25 n i m *-ma ri-ḫi* 9", "16 from 25 you lift: remains 9". Not knowing the fifth-century intermediate step, Otto Neugebauer took t a to be a genuine Sumerian suffix and translated "von 16 bis 25 steigst du auf, und es bleibt 9". Instead, the fifth-century text shows us that t a is nothing but a logogram for *ina*, the underlying phrase as a whole being Akkadian – with a reference to an operation on the reckoning board.

In consequence, the shift from one Sumerian term to another one must be explained not at the level of textual transmission or translation but as two instances of putting the same material operation into Sumerian words.

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[8] Thus the translation in [Sjöberg 1976: 173]; Castellino misses the mathematical point.

[9] All Mesopotamian dates are evidently BCE. For convenience, I follow the Middle Chronology where this distinction is pertinent.

The levels of terminology

After this warning that words – and in particular written words – are not the only instruments for, and not the only transmitters of knowledge, let us nonetheless turn to written words – first, and mainly, those used in Old Babylonian mathematics.

Such words belong at many levels. Restricting myself to what I am going to discuss, I shall list the following categories:

– First, there are *names for tools.* The "hand" was already mentioned, but tables are also tools. To the extent they can be shown to possess a name, they are clearly understood as such, not just as a collection of analogous items.
– Then there are *names for methods and tricks.* A delimitation of the range of variations covered by a particular name may be an important means for characterizing the type of mathematical thought within which they serve.
– Third, there are *terms and phrases used to structure* a mathematical text – for instance, to indicate that it constitutes a problem, and to delimit the various steps in the presentation and solution of a problem.
– Fourth, there are *names for mathematical objects,* also informative in different ways, not least when they conflate what for us seems to be different objects.
– Fifth and finally, there are *terms for mathematical operations.*

Names for tools

Old Babylonian mathematics made amply use of the tables connected to place-value computation; some uses – first of all of the multiplication table – are only implicit.[10] But occasionally the texts refer explicitly to i g i . g u b constants, and the reciprocals they "detach" (*paṭārum* / d u $_8$)[11] almost invariably appear in the standard table of reciprocals. In Old Babylonian problems about *igûm* and *igibûm*, "the reciprocal" and "its reciprocal", these are also pairs that

[10] The term a . r á is used repeatedly in AO 8862, but most of the multiplications spoken of thus are not found in the tables. We may conclude that it just stands for the multiplication of a number with a number, as it *also* does in the multiplication table.

The phrase *A.* e *s* í b . s i $_8$, used in many tables of inverse squares [MKT I, 70*f*], similarly appears in many problem texts, but again often in cases that are not listed in the tables. It thus cannot be taken as a reference to the table but only as a phrase shared with these.

[11] Here and everywhere in the following I make use of the "standard translations" used in the "conformal translations" of [Høyrup 2002a].

appear in the standard table (5 and 12, 1 30 and 40, 1 4 and 56 15, 1 40 and 36, 1 20 and 45, 1 12 and 50, 2 and 30, 1 40 and 36).[12]

However, these are nothing but references to items from the tables, and not to the tables as entities as such. Nor do the tables themselves carry titles. However, one problem text carries an explicit reference to a table.

This is the text BM 85200 + VAT 6599, famous for treating sometimes irreducible cubic problems about a parallelepipedal "excavation" (alongside a number of problems of the first and second degree about the same configuration – see [Høyrup 2002a: 137–162]). In rev. I 23, where 4`12 is to be factorized as $p \cdot p \cdot (p+1)$, we find "*i-na* í b . s i$_8$ 1 d a ḫ . ḫ a 6 [˹erasure?˺] í b . s [i$_8$]". In this construction, the first í b . s i$_8$ cannot be a verb, as everywhere else in the text, and d a ḫ ("to append") can never go with the preposition *ina*, "from". The only grammatically coherent interpretation is that *ina* governs the whole construction "í b . s i$_8$ 1 d a ḫ . ḫ a", which must then mean something like "equalside, 1 appended". The whole phrase thus means "from 'equalside, 1 appended', 6 is equal". Tabulations of $p \cdot p \cdot (p+1)$, which would correspond perfectly to the name "equalside, 1 appended", have indeed been found – see [MKT I, 76*f*] and [Friberg 2007: 56–58].[13]

Beyond that, some edubba texts refer to familiarity (or faulty familiarity) with the multiplication table; it is *identified simply as a . r á , that is, by means of the operation term appearing explicitly or implicitly in each line – see [Friberg 2000: 152].

Since these table types carried a name, others probably also did. But these have not made it into the written texts (at least not those that have been read and interpreted).

A term connected to tables is *nadānum* / s u m , "to give". The short text YBC 6295 tells how to proceed when "it does not give to you" (*la id-di-nu-kum*) the cubic side of a number – see [Høyrup 2002a: 65]. In general, the term is mostly used for the outcome of calculations in the place-value system (one text groups applies it more generally, see below); an origin in Ur III calculation is not implausible.

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[12] [MCT, 129*f*], [MKT I, 197, 346–349], [Friberg 2007: 252–254].

[13] VAT 8521 has a parallel reference to b a . s i . 1 . l á , "equalside, 1 diminished" ([MKT I, 352], cf. [Friberg 2007: 1]). Whether the interest asked for is meant to be listed within a table $n \cdot n \cdot (n-1)$ carrying this name or just to conform to this expression is unclear, however.

We might expect "giving" to be coupled to "taking", and while the side of a square is normally stated as "what is equal" (í b - s i$_8$ functioning as a verb) or "what is the equalside" (í b . s i$_8$ functioning as a noun), a few texts do "take" (*leqûm*) the equalside – thus Db$_2$-146, YBC 4675 and YBC 4662–4663. However, whether this is really meant as "taking" from a table is highly dubious. A number of texts "take" a fraction *of something* (whether determined as i g i *n* or with an ordinal; yet *a reciprocal,* as occurring in the table, appears never to be "taken" but to be invariably "detached" [*paṭārum* / d u $_8$]). Particular striking is TMS XXV, which in rev. 6 and 9 "takes" the third (*šaluštum*) of 30 (which would not appear in any table), but "detaches" i g i 40 and i g i 30 in obv. 3, 5 and 9. So, even though values are *given* by tables, they seem not to be *taken* from them.

Names for methods and tricks

Two methods are mentioned by name in the problem texts. One is the *makṣarum*, derived from *kaṣārum*, "to bind together"; it may thus be translated "bundling". It occurs in three texts. The first is YBC 6295, just mentioned, which explains what to do when the cubic side of 3°22′30″ is not "given".[14] The method is to subdivide this volume into volumes 7′30″, of which there turn out to be 8. We may see the subdivided cube as a "bundle" of 2×2×2 smaller cubes, and the initial line of the text states indeed that what follows is the "bundling of a (cubic) equilateral".

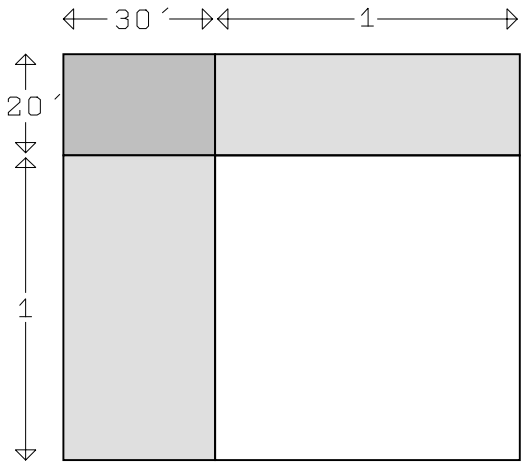
The second text mentioning the method is YBC 8633 [Høyrup 2002a: 254]. Here, a triangle with width 20 and longest length 1`40 is supposed to be subdivided into smaller triangles with sides 3, 4 and 5 (since the original triangle is far from being right, this is not possible, but that is immaterial for the present discussion). The requested factor 20 is spoken of as "the bundling"; but in a heuristic summary the whole procedure is also spoken of as the "bundling of a trapezium (s a g̃ . k i . g u d) with cross-over (*ṣiliptum*, i.e., diagonal)".

The third occurrence of the term is in the Susa text TMS XVII. The text is damaged, but here it appears to have to do with the partition of an area (the square on the sum of the sides of a rectangle) into sub-areas.

The other procedure spoken of by name turns up in the Susa text TMS IX, section 2 [Høyrup 2002a: 90–93]. This didactical text explains how to transform

[14] When orders of magnitude are not freely chosen, I use Thureau-Dangin's notation (not his invention, indeed, but used by other Assyriologists at least since 1911) to indicate an adequate absolute order of magnitude. In the present case, 3°22′30″ might be replaced by 3‴22‴30⁗ (etc.) but not, for example, by 3′22″30‴.

the sum of *the area*, *the length* and *the width* of a rectangle (⊏⊐(ℓ,*w*)+ℓ+*w*, ℓ = 30′, *w* = 20′) into a rectangular area "by the Akkadian (method)", *i-na ak-ka-di-i*. At first, ℓ is replaced by the rectangle ⊏⊐(ℓ,1) and *w* by ⊏⊐(*w*,1). This generates a quasi-gnomon, a rectangle from which a square □(1) is lacking in a corner (see the diagram). "Appending" this square we obtain a rectangle ⊏⊐(ℓ+1,*w*+1). After verifying that this rectangle fulfils the conditions, the explanation closes with the words "thus the Akkadian (method)", *ki-a-am ak-ka-du-ú*.



Section 1 of the same text explains the trick of transforming ⊏⊐(ℓ,*w*)+ℓ into ⊏⊐(ℓ,*w*+1). This trick has no name. What is new in section 2 is thus the quadratic completion, albeit an idiosyncratic variant – actually not found anywhere else in the corpus, even though texts exist where it *could* easily have served (e.g., AO 8862). That a name should be reserved for a method that occurs in a single text only (furthermore of late Old Babylonian date) is unlikely. It seems reasonable to assume that it refers to the method of quadratic completion in general, the normal type as well as whatever variants might turn up.

The *makṣarum*, we saw, also designated not a single procedure but a spectrum of (not too closely) related methods. According to the philological principle "Once is never, twice is always"[15] we may guess that this flexibility (or, if preferred, fuzziness) characterized the general view of Old Babylonian calculators of their panoply of methods.

Structuring terms and phrases

Restricting ourselves to mathematical texts proper (that is, omitting accounting and corresponding *uses* of mathematics), the corpus can be divided into three text types: tables; tablets for rough work; and problem texts – in didactical order, cf. [Proust 2008], tables being trained and learned by heart before being applied in elementary calculations, and problem texts being apparently a matter for specialists, outside the normal full curriculum (as we know it not least from Nippur) but presupposing it.

[15] My thanks to Eckhard Keßler for this jibe, which may go back to Ulrich von Wilamowitz-Moellendorf [Kahn 2003: 350].

Tables were structured spatially, but apart from the words appearing in the single lines (a . r á , i g i g á l , etc.) not by means of words. Tablets for rough work are less uniform. Very often they contain numbers only – many examples are in [Robson 1999: 247–277]. But they *may* carry numbers as well as a geometric diagram – the most famous example being YBC 7289, which determines the side of a square by means of an i g i . g u b value.[16] Finally, they may border the category of problem texts, and contain a question marked . b i e n . n a m ("its ... what?") and a possessive suffix . b i ("its") glued to the answer, as in the Nippur texts UM 29-15-192 and Ni 18, as well as CBS 11318.

e n . n a m is an innovation (already found in the few late 19th or early 18th-century problem texts from Ur, see [Friberg 2000: 139–144] and below), but use of . b i to mark a question or the quantity that is found goes back to Early Dynastic and Sargonic school texts [Powell 1976: *passim*; Foster & Robson 2004, *passim*]. Direct continuity is not to be expected, however: the Sargonic texts regularly use the verb p à d (= p à), "to see" or the allograph p a for results found or to be found; this is totally absent from the Old Babylonian problem-close tablets for rough work (but not from all genuine problem texts, see below).

Problem texts: the text groups

Before we proceed with the discussion of the structuring of problems, a presentation of the groups into which these fall will be adequate.

A division of the Old Babylonian corpus into a "southern" and a "northern" group was first proposed by Neugebauer [1932: 6*f*]. It was elaborated by Albrecht Goetze [1945], who based his analysis mainly on orthography but also to some extent also on vocabulary (not terminology, since he did not take differences of meaning into account). Goetze divided the corpus of problem texts as known by then into six groups.

At a time when Assyriologists tended to regard texts containing too many numbers, in particular too many sexagesimal place-value numbers, as a "matter for Neugebauer" (who wrote his last paper on Babylonian *mathematics* together with Abraham Sachs in 1951 – mislaid but eventually published as [Neugebauer & Sachs 1984]), and during which most historians of mathematics still thought in terms of perennial "Babylonian mathematics", Goetze's analysis had little impact.

[16] Three more – YBC 7290, YBC 11126 and YBC 7302 – are published in [MCT, 44], and nine in [Friberg 2007: 189–204].

In 1996, having been invited by Hans Neumann to contribute to the Oelsner-Festschrift with the page limit "schreib so viel du willst!", I took up the matter where Goetze had left it, including now the texts groups from Ešnunna and Susa, which had not been known in 1945, and looking more specifically at terminology and structuring phrases (the paper was published as [Høyrup 2000]). With minor exceptions my analysis confirmed Goetze's division and Neugebauer's original hunch while adding the two new text groups. After the appearance of Jöran Friberg's study of the texts from early Old Babylonian Ur I included a revised version as chapter IX of [Høyrup 2002a], on which I draw heavily and mostly without specific references in the following.

According to this new analysis, the corpus of Old Babylonian problem texts falls into the following groups (I use Goetze's numeration as extended in [Høyrup 2000] and [Friberg 2000])

1. According to Goetze "certainly to be localized in the South, in all probability Larsa".
2. According to Goetze "likewise a southern group". The important theme text BM 13901 has to be eliminated from the group; what remains may be designated "2A".

ii. The single tablet BM 13901, which Goetze had placed in "group 2" for reasons which he himself characterized as circular, and which can now be seen to be irrelevant – but the text is certainly also southern.

3. According to Goetze localized in Uruk.
4. Linguistically indistinguishable from "group 3. Its "provenience may likewise be Uruk".
5. Considered unspecifically northern by Goetze, and consists of only three texts, one of which is a fragment and one heavily damaged. For terminological reasons, Eleanor Robson [2001: 183] proposes at least the third, YBC 6967, to belong to "group 4"; but it shares as many terminological features with Haddad 104 (from Ešnunna, "7B"),[17] for which reasons the matter is best left pending.
6. Considered by Goetze to combine "northern and southern characteristics" and to be "slightly younger in date than the other groups". A footnote intimates a connection to Sippar, which has since then been corroborated and may now be considered fairly well-established.
7. Regularly excavated texts from Ešnunna. A subgroup "7A" consists of terminologically very similar texts found within neighbouring rooms; the remainder "7B" has no inner coherence and is only considered a "group" for convenience. Most texts are found in dated contexts (1790 to 1775).
8. Regularly (but rather badly) excavated texts from Susa, probably of late Old Babylonian date.

[17] In particular the results of calculations "coming up", cf. below, which they never do in "group 4".

Ur. Regularly excavated texts (but many found as fill) from 19th or early 18th-century Ur.

S. "Series texts", which Goetze did not consider because they contain almost no syllabic Akkadian. Neugebauer, who was the first to discuss the group [MKT I, 383*f*], proposed it to be from Kiš, but gave up the idea (as well as the term) in [MCT, 37]. They carry the name because the single tablets indicate in a colophon to be number so-and-so of a series.[18]

Problem formats and history

Taking into account a combination of external and internal criteria, we may construct a plausible scenario for the development of the Old Babylonian culture of mathematical problems.

The "Ur group" contains a few genuine problems only. Moreover, these exhibit no thematic intersection with what we find in the later Old Babylonian groups, and the problem format is rudimentary – a question e n . n a m (a . n a . à m if an accusative is required) and an occasional ì . p à d . d è , "you will see" or a suffix . à m , "it is" indicating a result [Friberg 2000: 139–144, *passim*]. We seem to be at the watershed where the culture of problems is emerging, but still on the sole basis of the Ur III tradition.[19]

The earliest member of "Group 7", IM 55357 from c. 1790, already has a more developed structure. After presenting the data it asks an explicit question; the prescription is introduced by the phrase z a . e a k . t a . z u . u n . d è , "You, to know the proceeding". Questions are asked by a syllabic *mīnum* or (in one place where an accusative is needed) a . n a . à m .[20] Results are "seen", but the phrase is i g i . d ù (unorthographic for "open the eye", that is, "see"). The semantics is the same as in the "Ur group" and the Sargonic problems, but there is

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[18] Friberg [2000: 264] suggests to move the texts VAT 7528, YBC 4669, YBC 4698 and YBC 4673 ("Gruppe C" according to [MKT I, 506]) to a subgroup "2B" belonging together with "2A", the expurgated "group 2". Apart from the absence of serial numbering from the "2A" catalogues there indeed are outspoken similarities. These four texts also do not exhibit the complex organization of the other series texts described below. Since incipient serialization was a general phenomenon in late Old Babylonian scribal culture, serialization of mathematics may indeed have started in different places.

[19] The absence of a culture of mathematical problems in Ur III is dealt with in [Høyrup 2002c].

[20] This term, we remember, was also used in a single text from Ur. It seems never to turn up elsewhere. The outspoken differences in other respects seem to exclude that the Ešnunna text was inspired directly by what went on in Ur.

obviously no direct continuity at the terminological level. We must presume, either that already the Sargonic texts translate an Akkadian term (*tammar*), or that Sumerian p à d has first been translated into and transmitted in Akkadian and then retranslated into Sumerian, the retranslator accidentally choosing a synonym.

The writing makes heavy use of logograms, for which reason it is impossible to ascertain whether the later systematic change of grammatical person (see imminently) was intended.

The texts from subgroup 7A, published in [Baqir 1951], share a new feature: an opening phrase *šum-ma ki-a-am i-ša-al(-ka) um-ma šu-ú-ma*, "If [somebody] asks (you) thus:" This is the typical opening of a riddle, and reveals an important source for the problem culture of the Old Babylonian scribe school – namely the professional riddles of mathematical practitioners (mostly but not exclusively surveyors). The statement itself is then mostly formulated in the first person singular ("I have [done so and so]").

The prescription opens with the formula *at-ta i-na e-pé-ši-ka*, "you, by your proceeding" – close to that of the early text IM 55357, but now in syllabic Akkadian. Its "you" is followed up by use of the present tense, second person singular.

Often, the transition to a new section of the prescription is marked by the phrase *na-ás-ḫi-ir*, "turn yourself around".

Results of calculations are marked by one of the phrases *ta-mar*, "you see", or *i-li-a-ku-um*, "comes up for you" – in both cases often combined with an enclitic *-ma* on the verb for the operation.

A strange feature, with no analogue elsewhere in the corpus, is a coupling between interrogation and the announcements of results: when results "come up", the interrogative phrase of the question is *mīnum*, "what"; when they are "seen", we find *kī maṣi*, "corresponding to what". Possibly, two scribes with different habits were at work.

As stated, "7B" is no group proper. Its eight members come from various locations – Tell Ḥarmal, Tell Dhibaʾi, and Tell Ḥaddad. However, most of them open prescriptions by some variant of the phrase "You, by your proceeding" – one has a simple "You". Prescriptions carry the closing formula *kīam nēpešum*, "thus the procedure" (in contrast to group 7A).

Two texts open as riddles, "if somebody ...". Haddad 104, containing 10 problems about topics rooted in Ur III practice, opens the statement *nēpeš*, "procedure of", or (if a variant is announced) *šumma*, "if [however]". IM 52301

opens the statement *šumma*, the early IM 55357 by stating the object (s a ĝ . d ù , "a triangle"), and IM 121613 by describing the situation.

Transitions to new sections may be marked by *ta-ás-sa-ḫa-ar*, "you turn around"; *tu-ur* or *tu-úr*, "turn back"; or as in "7A" *na-ás-ḫi-ir*, "turn yourself around".

Results may be "seen", or they may "come up for you".

All in all, the Ešnunna texts reveal conscious attempts to create a problem format, but obviously no agreement about how this format should look; only "7A", presumably reflecting the ways of a single teacher or team of two teachers, has achieved something systematic. This, as well as the frequent riddle format, shows that we are confronted with the early phase of the development of a tradition[21] – which was then interrupted when Hammurabi conquered and destroyed the Ešnunna state in 1761. In spite of this, it is striking that most of the favourite themes of Old Babylonian mathematics are already dealt with.

Hammurabi *may* have brought Ešnunna scholars back to Babylon; in any case, the relation between the Ešnunna and the Hammurabi law codes indicate that he brought inspiration. No less hypothetical is the possibility that he brought back teachers of mathematics. In any case, the Old Babylonian strata of Babylon are covered by later remains.

What we do know is that the problem culture turns up soon afterwards in the south. An important text belonging to "group 1", the prism AO 8862, is obviously related to a prism carrying tables in the Ur-III tradition (metrological tables and tables of squares, inverse squares and inverse cubes) that was written in Larsa in 1749 [Proust 2005]. Vacillating conventions (but mostly concerning the terminology for operations) both within this text (and within other texts from the same group) and between texts bwlonging to the group suggest that this group also reflects an incipient, not a mature tradition.[22]

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[21] In a similar vein, Jean-Jacques Glassner [2005] uses the inhomogeneity of the technical terminology of haruspicy as evidence of a still immature discipline.

[22] A beginning around 1749 is contradicted by Eleanor Robson's dating [2001: 172] of the tablet Plimpton 322 (which she supposes to be from Larsa) to "the 60 years or so before the siege and capture of Larsa by Hammurabi of Babylon in 1762 BCE", Her argument, however, is far from coercive. She observes that the tablet is in landscape format, and that this format was used in the Larsa bureaucracy from 1822 onward. However, the contents of the text – a table with many columns – asks for this format. Even if it had gone out of administrative fashion after the conquest, it would be an obvious choice to use it when it was adequate – the particular southern spelling of the mathematical texts show that they were written by scribes who had received their education locally.

As a rule, the texts belonging to the group open by stating either the object or the situation. The prescription normally opens with an Akkadian syllabic "you, by your procedure" or "by your procedure"; there is no closing formula. Results are mostly marked by nothing but an enclitic *-ma* on the preceding verb, but sometimes they "come up".

The riddle introduction has disappeared. In groups 2–6 and 8, where it is also absent, the system of two voices is reinterpreted, and the statement stands out as if is was formulated by the master telling the situation "I" have produced, while the prescription is formulated by the instructor or "elder brother"[23] in the second person singular or the imperative, at times arguing for a particular step with an exact quotation of what "he" (the master) has said. This system is still only imperfectly present in group 1, where the prescription may shift between what "you" shall do and what "I" do, and where results sometimes come up "for me" and sometimes "for you" (regularly within the same text). This fits an incipient, still not firmly established tradition.

Striking is the absence of *tammar*, "you see", not only from this group but also from the other southern groups (2–4). Since this term was characteristic of the Ešnunna texts and presumably of the Akkadian lay (non-scribal) tradition, avoiding it[24] may have been a way to demarcate oneself from the conqueror.[25]

The core of "group 2A" (the expurgated "group 2") is constituted by two theme texts about "excavations", to which come a number of statement catalogues without prescription – in part containing the statements of the theme texts, and thus certainly coming from the same locality and school. The statements (of theme

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[23] This š e š . g a l is a familiar figure from the *edubba* literature. Cf.. e.g., [Kramer 1949, *passim*].

[24] An oblique reference in the "group 3" text YBC 4608 (a question what to do *aš-šu X a-ma-ri-i-ka*, "in order that you see *X*", shows that the idiom was known. YBC 4662, belonging to "group 2A", also has a single isolated *tammar*. The almost complete but not total absence of *tammar* must thus reflect a conscious effort to avoid it.

[25] This argument does not presuppose any kind of patriotic feelings, which may or may not have been there. A local elite will automatically resent coming under control of foreigners and thus to descend the hierarchical ladder – as pointed out sharply by Samsî-Addu to his son Yasmah-Addu deputy king of Mari when the latter had expressed the intention to give official functions to captive nobles from Ešnunna [Durand 1997: I, 182*f*].

This was probably more than the mere suspicion of a cautious and shrewd ruler. Michel Tanret [2010: 247] points to a symbolic act of resistance on the part of a temple manager in Sippar against the Babylonian conqueror.

texts as well as catalogues) are heavily logographic, the prescriptions of the theme texts predominantly syllabic.

The statements start by announcing the situation, and then ask a question marked e n . n a m , "what" (in a single case *kī maṣi*, "corresponding to what"). The logographic phrase z a . e k i d$_9$ / k i d . d a . z u . d e ,[26] "you, by your making", serves to open the prescription. In one of the theme texts it closes *ki-a-am né-pé-šu*. As a rule, results "come up for you" (but as mentioned, the theme text YBC 4662 contains a single *tammar*).

Many problems in the theme texts combine the determination of the geometric object (which may constitute a directly geometric or an "algebraic" problem[27]) with a calculation of the wages to be paid, thus with a normal scribal concern. This, as well as the format, suggests that the texts of this group constitute a direct continuation of the normal mathematical curriculum of the scribe school.

The linguistically indistinguishable "group 3" and "group 4" are probably both from Uruk. None the less, they are different in their choices of format and even more as terminology is concerned – so different that one may suspect deliberate demarcation. Internally, each group is rather coherent.

In "Group 3", the statement is an unadorned presentation of the situation, ending with a question (mostly marked e n . n a m , more rarely *kī maṣi*, once in a problem about the distribution between brothers *kiyā*[28]). If a prescription is present, it opens with the phrase z a . e k ì d . d a . z u . d e . There is no closing formula.

Results of any kind are followed by a logographic s u m , "it gives" – except in four passages, where it is syllabic. Three are instances of the "division

[26] Both unorthographic, which (like the isolated occurrence of *tammar* in YBC 4662) is perhaps evidence of use of northern material – orthographic writing would have employed k ì d .

[27] I shall abstain from taking up the question whether Old Babylonian "algebra" is justly characterized as an algebra or not, which others find much more interesting than I do, and the answer to which depends on definitions and taste. Examination of the discipline in question (which I shall go on referring to in quotes) is the main topic of [Høyrup 2002a] as well as [Høyrup 2010].

[28] In the Old Babylonian corpus, this is the normal way in all groups to ask for several values, and it may thus adequately be translated "how much each". Non-mathematical contexts appear not to ask for this plurality [CAD 8, 329a].

question", "what shall I posit to *P* which gives me *Q*?"; the syllabic writing thus serves to make clear that a subjuntive is meant.

The only "logical operator" appearing in the group is *aššum*, "since"; it introduces an argument by "single false position" in VAT 7532 and VAT 7535, "since ⅙ of the original reed was broken off, inscribe 6, let 1 go away, ...".

"Group 4" also opens statements by describing the situation – occasionally defining the object first; the question is made explicit, mostly by e n . n a m (in a peripheral subgroup by a syllabic *mīnum*), more rarely by *kī maṣi*, and in one "brother problem" by *kiyā*. In a few cases the prescription starts by *atta*, "you", but mostly there is no opening phrase, as there is no closing formula.

Results are mostly marked by a preceding enclitic *-ma*. Syllabic writings of *nadānum*, "to give", are mostly used in connection with the "division question", but on a few occasions for the outcome of "raising" multiplications.

The logical operator *šumma*, "if", is used regularly, sometimes in the beginning of statements regarded as variants (which excludes its being a remnant of the "riddle opening"), more often in the beginning of final verifications, which are frequent in this group. In three texts it is used within the prescription to open a new line of reasoning after a preliminary result has been established. *aššum*, "since", is used to introduce quotations from the statement, and furthermore once in a broken, incomprehensible passage (VAT 8523, rev. 8).

In contrast to "group 1", the two Uruk groups look as if they represent already settled local traditions. Uruk and Larsa being separated by less than 25 km, it is rather unlikely that they can have produced before a "Group 1" still groping for a canonical style. A date after c. 1740 seems inherently more plausible.

At the same time, it is virtually certain that all southern texts (groups 1–4) were produced before 1720 – after the successful secession of the Sealand there seems to have been a violent decline in literate culture in the area.

Whether the three texts counted as "group 5" really form a group is uncertain, as is its localization in the North – cf. above, p. 10. In any case, the "group" is too small to tell us very much. "Group 6" is much more interesting.

It is certainly northern, and in all probability to be located Sippar. In all probability it is also later than the southern groups. To its core ("6A") belong a number of procedure texts containing many problems, one (BM 85200 + VAT 6599) strictly dealing with "excavations", see above, others (BM 85194, BM 85196, BM 85210) either "theme texts" with a very liberal idea of how to delimit

the theme (“geometrical calculation of anything”?) or outright mathematical anthologies.

As a rule, “you see” results in this group, which shows it not to descend from the southern groups but to be a later member of the same extended family as the Ešnunna texts.

Very often, statements start by defining the object. Sometimes, however, this is omitted, and we get a description of the situation (often neutral, but at times in the first person singular). In a few cases, mostly not concerning variants, the beginning is *šumma*, “if”. This *might* be a remnant of the riddle opening, but nothing else in the texts supports such a connection. On a few occasions, the statement is supported by an explanatory diagram. The question is normally asked with e n . n a m , very rarely with *kīmaši*. Prescriptions open z a . e , “you”, and close *nēpešum*, “the procedure”, occasionally *kīam nēpešum*, “thus the procedure”.

Beyond *šumma*, the logical operators *inūma*, “as”, and *aššum*, “since”, both turn up a few times, the former to introduce an embedded small piece of reasoning, the second probably with the same function (but all relevant passages are strongly damaged).

kīam nēpešum, “thus the procedure”, was also used in “7B” as the closing formula. It is totally absent from the southern texts, also in the abbreviated form *nēpešum*. This corroborates the conclusion derived from the use of *tammar*, namely that “group 6” belongs to the same family as the Ešnunna texts. There is no reason to believe that its style was borrowed from scholars who had emigrated from the Sealand.

The “series texts”, on the other hand, or at least some of them, may be in debt to southern scholars, even though their almost certainly late date[29] tells us that they must have been produced in the North.

As stated above, the single tablets indicate their number within a series; partial overlaps etc. shows that several such series existed, and that there is no trace of “canonization”.

The texts contain only problem statements (and sometimes a numerical answer). They are written in a very compact and highly stylized logographic

[29] Firstly, the utterly compact formulation of these texts must be the outcome of a long development; secondly, serialization seems in general to have taken its beginning in the final Old Babylonian century. Finally, Christine Proust [2010: 3, cf. 2009: 195] argues from the structure of the colophons for links to “a tradition which developed in Sippar at the end of the dynasty of Hammurabi”.

notation – even prepositions are replaced by Sumerian case endings, but none the less the language is even farther from being Sumerian than Akkadian.[30]

Within the single statements, there is no problem format apart from a facultative e n . n a m specifying the question. Globally, however, only the existence of a strict format allowed the users of the text (and allows us, when we are lucky!) to understand the situation that is delineated. As an example, we may look at the translation of a sequence from YBC 4668 (following [Høyrup 2002a: 201*f*]). Round brackets (...) are explanations that are needed for minimal comprehensibility, pointed brackets ⟨...⟩ indicate words that according to the general style of the text should have been there but are none the less omitted.

Rev. III

#34 **4.** The surface, 1 e š è
5. The fraction, of the width, concerning the length
6. to the length raised, 45.
7. The fraction, of the length, concerning the width,
8. ⟨to⟩ the width raised: 13°20´
9. its length, width what?
...
#38 **19.** The 19th part (the excess) of (that) which to the length (is) raised
20. over (that) which to) the width (is) raised, goes beyond
21. (to that) which to the length (is) raised, appended, 46°40´.
#39 **22.** (In) steps 2 repeated, appended, 48°20.
#40 **23.** Torn out: 43°20´.
#41 **24.** (In) steps 2 repeated, torn out, 41°40´.
#42 **25.** (To that) which to the width (is) raised, appended: 15.
#43 **26.** (In) steps 2 repeated, appended: 16°40´.
#44 **27.** Torn out: 11°40´.
#45 **28.** (In) steps 2 repeated, torn out, 10.
#46 **29.** The surface, 1 e š è
30. The 7th part of (that) which (to) the length, (of the) width, (is) raised,
31. (and that) which (to) the width, (of the) length, appended, 53°20´.
#47 **32.** (In) steps 2 repeated: 1`1°40´.
#48 **33.** Torn out: 36°40´.
#49 **34.** (In) steps 2, torn out, 28°20´.
#50 **35.** (To that) which to the width (is) raised,

[30] Neugebauer [1934: 70–72] compares this compact logographic writing to an algebraic symbolism, though explaining how this has to be understood in order to be adequate – actually, his interpretation looks more like the description of an algorism than as a manipulation with symbols. Later general histories of mathematics have sometimes been too eager to claim the algebraic symbolism without caring for Neugebauer's restrictive use of the term.

36. appended: 21`40´.
#51 **37.** (In) steps 2 repeated, appended: 30.
#52 **38.** Torn out: 5.
#53 **39.** (In) steps 2 repeated:
40. 3°20´ it went beyond.

All problems deal with a rectangle ⊏⊐(ℓ,w). In #34, we are told that the area ⊏⊐(ℓ,w) is 1 e š è , that is, 600 times the square on the basic length unit. Further, $L = {}^{\ell}/_{w} \cdot \ell = 45$, $W = {}^{w}/_{\ell} \cdot w = 13°20´$. Since ⊏⊐($L$,$W$) = ⊏⊐($\ell$,$w$), this is a standard problem, just embedded in a rather trivial complication.[31] In #38, the area is presupposed to be unchanged (whence not mentioned); the other condition is ${}^{1}/_{19} \cdot (L–W)+L = 46°40´$. The single line in #39 means that the second condition is now ${}^{2}/_{19} \cdot (L–W)+L = 48°20´$, while that of #40 is $L–{}^{1}/_{19} \cdot (L–W) = 43°20´$. In #42, it becomes ${}^{1}/_{19} \cdot (L–W)+W = 15$, while #46 changes the denominator from 19 into 7. We thus have exploration of all the possibilities obtained by changing the second condition along 4 dimensions.

Grammatical, medical and extispicy lists also attempt to be systematic, and sometimes we find sequences which vary along two dimensions – but not more, and never as perfectly as here, for the simple reason that the subject-matter does not allow it.[32] Within mathematics, we also have nothing coming close, neither earlier nor later. Even more obviously than the Old Babylonian problem culture in general, the series text represent a species that was too highly specialized to survive the particular environment where it had emerged.

We know about selective adoption / adaption of Mesopotamian metrology and tables outside the Babylonian area in the second and first millennium, but the only place where we have evidence of a broad adoption of the *problem culture* is in (probably late Old Babylonian) Susa.

The texts contained in the volume [TMS] are evidence of that – more precisely "group 8A" consisting of the procedure texts TMS VII–XXV.[33] For the present purpose the most important observation to make is that results are marked *tammar*, "you see". Statements open by describing the situation which "I" have

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[31] Once L and W are found, we have to use that ${}^{L}/_{W} = ({}^{\ell}/_{w})3$. Knowing ${}^{\ell}/_{w}$ and ⊏⊐(ℓ,w) we find □(ℓ) as their product, whence also ℓ – etc.

[32] Nor are all series texts as systematic as this passage. The variations in for instance YBC 4714 [Høyrup 2002a: 112–132] are no more orderly than those of well-structured medical texts.

[33] TMS V–VI, "group 8B", are two statements catalogues, ""8C" a single atypical procedure text. TMS I–IV are tables and drawings of polygons with numbers written into them.

created. The prescription opens with a simple *atta*, "you", and ends (except in the two texts that explain that this was the "bundling" or "Akkadian" method, cf. above) just by pointing out that the number resulting from the last calculation is the quantity asked for.

The appearance of *tammar* shows that the Susa texts belong to the same broad area as the Ešnunna- and Sippar-texts. Other characteristics make it clear that they do not descend directly from any of these particular groups. If it is true, as argued above, that the southern traditions were only established after c. 1750 and vanished before 1720, we should perhaps not wonder that they did not leave a strong impact in the following century. The good luck of excavators – that cities burn or are left in haste – is not necessarily the best for the influence of scholarly traditions.

Names for objects

There is no reason to discuss the names for objects with respect to the single text groups – to a large extent they are used transversally. I shall restrict myself to two observations.

One has to do with a tendency to apply "default understanding". If a problem statement presents its object as u š s a g̃ , "length width", it does not deal with a length and a width but with a figure characterized by possessing a length and a width – and moreover, by the simplest figure (as seen by the Babylonians) which is characterized by possessing them, that is, a rectangle. And when Db_2-146 starts *šum-ma ṣi-li-ip-ta-a-am i-ša-lu-ka*, "if about a cross-over (somebody) asks you", the meaning is that he asks about the simplest possible configuration possessing a cross-over (i.e., diagonal).

Both expressions reflect the fundamental way of the Babylonians to think of the objects of their mathematics – namely "by default".

This does not correspond to what we believe about our own thinking – but perhaps we are wrong, and perhaps we are more Babylonian than we recognize. In languages where the counterparts of "quadrilateral" (German *Viereck*, Danish *firkant*, and even French *quadrilatère*, Arabic *murabbaʿ*) belong to current non-technical speech, they are often used in the more specific sense of *square* (in Arabic even primarily). Before the monster-hunt of nineteenth-century mathematicians, a *function* was also presupposed to be not only continuous but also smooth. If it was not, that had to be made explicit. And much of the argument in Lakatos's *Proofs and Refutations* [1976] is indeed built up around objects that are gradually discovered not to possess necessarily the properties (convexity etc.) that were presupposed by default.

The other observation to make has to do with syllabic versus orthographic writings of the terms for "length" and "width". As we have already seen, the "you" introducing a prescription is written z a . e in some text groups and *atta* in others. Many terms for operations behave similarly, or they are written logographically in straight and syllabically in oblique forms (for instance, the subjunctive and the precative); a logographic term in a statement may even be quoted syllabically as what "he" has said.

However, the case of "length" and "width" is different. We may start by observing that they occur in two different roles. They may be the extensions of *real* geometric objects – a carrying distance, the length of a wall, or the dimensions of a real field. In that case they may be written either way, as *šiddum* / u š (“length”) respectively *pūtum* / s a g̃ (“width”). But they may also be the dimensions of the abstract rectangles used as a basic representation in the "algebra", and then they are invariably written logographically, and without any grammatical complement that might indicate an Akkadian pronunciation – *except* in a few texts from Ešnunna and two texts plausibly from early Sippar.[34] It thus seems that a firm conceptual distinction between real distances and the "abstract" extensions used in "algebraic" representation was only establishing itself around 1775. In all later text groups its presence is subject to no doubt.

Operations

Even the terminology for operations need not be systematically discussed with respect to the single groups – this would not yield much further information, nor contradict the results already obtained. Instead, a list of operations and corresponding terms will do, with observations about their occurrence when such are called for.

Additive operations

One addition consists in joining one magnitude *d* to another one *A*. In this process, *A* conserves its identity but increases in magnitude; the sum thus has no name of its own. The operation is concrete, and *d* and *A* must by necessity be of the same kind. The Old Babylonian term for this operation is *waṣābum* (I use the standard translation "to append"). In two texts from the disparate "group 1" (YBC 6504 and AO 6770), in "group 3", "group 6", and in the series and Susa texts it may be replaced by d a ḫ (sometimes with grammatical complements).

[34] The "Tell Harmal Compendium" (IM 52916+52685+52304), Db_2-146 and IM 43993; and CBS 43 and CBS 154+921, which indicate syllabic possessive suffices, cf. below, note 47.

The word is regular Sumerian, meaning “to add, to say further, to help” [Thomsen 1984: 298]; the idea to use it as a logogram for *waṣābum* seems to be secondary.[35] The inverse operation of this addition is subtraction by removal, see presently.

Another addition is *kamārum*, “to accumulate” (or “to heap”). It is symmetric, and dissolves the two addends into a common sum (*nakmartum* or, if still understood as the plurality of constituent parts, the plural *kimrātum* of *kimirtum*). It may be used for the formal addition of quantities of different kinds, in which case the addition really concerns the measuring numbers of the quantities involved. The logogram g̃ a r . g̃ a r appears to be of genuine Sumerian origin, cf. above, p. 4. “Group 6” and the Susa texts instead use UL.GAR, which is unexplained. The rarely used inverse of this addition is separation into constituent components (*bêrum*).

In some early “algebraic” texts (from Ešnunna and “group 1”), sides of rectangles and squares are “appended” to areas, which implies that they are regarded as “broad lines”, provided with a standard width equal to one linear unit.[36] These appear to have been eliminated in the same process as established mature problem formats. Afterwards these additions were always formulated as “accumulations”.

Subtractive operations

There are two subtractive operations, but a whole gamut of terms for them. One if *removal*, the inverse of “appending” and equally identity-conserving; the entity that is removed has to be a part of the one from which it is removed. The main terms for this are *nasāḫum*, “to tear out”, and *ḫarāṣum*, “to cut off”. The latter is mostly used in the Ešnunna texts and in “group 1”. It may perhaps have been the preferred term of the lay surveyors, who provided the basis on which the “algebraic” discipline was developed; this assumption would agrees with the absence of a corresponding logogram. The former, which replaced it as the normal term for the operation (but is already found in three texts from Ešnunna), was provided with the semantically improper logogram z i , cf. above, p. 4.

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[35] A lexical list states d a ḫ to be the equivalent of *ruddûm*, “to add (numbers, silver, commodities, goods, immovable property), to add words, entries in a tablet, to add a statement” (<*redûm*) [CAD 14, 226*f*]. This word seems never to appear as a mathematical term.

[36] The frequent appearance of this conceptualization of lines (which allows lengths and areas to be measured in the same units) in pre-Modern practical geometries is discussed in [Høyrup 1995].

In situations where connotations suggest a different metaphor, other terms for removal turn up occasionally. One example was quoted above for a different purpose, namely "since $\frac{1}{6}$ of the original reed was broken off, inscribe 6, let 1 go away, ..." (VAT 7532, VAT 7535). "Let go away" translates *šutbûm* (<*tebûm*). In non-mathematical contexts, this verb is regularly used for removing something that *should* go away, which is exactly the case in these false-position arguments.

Another rare term for removal is *tabālum*. It occurs in the Susa problem catalogue TMS V, section 12, in which a part of an area is "withdrawn"; since this "area" might also be a real field, the problems could very well deal with the real-life situation where this part has been *withdrawn by legal action*, as is the normal use of the term.[37] The other is the text YBC 4608 (obv. 24), where a line *a* is withdrawn from what is already known to represent the sum *a+b* of two opposite sides of a quadrangle. Probably, the term is chosen here because of the connotation of something which is due to be done.

That connotations played a role is confirmed by those texts which employ *ḫarāṣum* and *nasāḫum* together (in particular AO 8862): they tend to "cut out" from lines and "tear out" from surfaces.

The other subtractive operation is *comparison* of different entities. Most often, it is made by the phrase *A eli B d itter*/*ītar*, "*A* over *B*, *d* it goes/went beyond" (from *eli ... watārum*, "go beyond", "be(come)/make greater than"),[38] with the Sumerographic equivalent *A* u g u *B d* d i r i g. In the Susa texts d i r i g also serves as a logogram for the excess, that is, for that amount *d* by which *A* "goes beyond" *B*.

However, various reasons may determine that the comparison is made the other way around.[39] Then the text does not say by how much *A* exceeds *B* but instead by how much *B* falls short of *A*, using the verb *maṭûm*, "to be(come) small(er)" (Sumerogram l a l).

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[37] This interpretation fits the fact that these statements are in the third, not the first person singular. It is not the teacher who is supposed to have performed the action, as in the other statements.

[38] If we give up the ambition to render the grammatical structure of the Akkadian phrase, we may also translate "*A* exceeds *B* by *d*".

[39] The systematic structure of series texts may be one such reason; another is the aspiration that relative differences should be one of the "favourite fractions" ($\frac{1}{4}$, $\frac{1}{7}$, $\frac{1}{14}$, etc.) and not for instance $\frac{1}{6}$ or $\frac{1}{8}$ [Høyrup 1993]; finally, the grammatical habit to take the outcome of one calculation as the subject of the next sentence may require that this outcome be said to fall short of another quantity.

"Multiplications"

Three genuine multiplicative operations can be found in the Old Babylonian texts. One is a . r á , "steps of", the multiplication of number by number. It is the term of the multiplication tables (including the table of squares) but very rare in problem texts.[40]

The second multiplicative operation is *nasûm* / í l (occasionally n i m), "to raise". Its origin is in volume calculation, and it refers to the "raising" of the base from the default height of 1 cubit to the real height; from there it was transferred to other multiplicative determinations of concrete magnitudes.[41] In particular, it is always used in multiplications by technical (i g i . g u b) constants and by reciprocals. We have no evidence that it was already used during Ur III, but on the other hand we have no texts where we would expect it to turn up. Since the result of a "raising" is often stated to be "given", also in groups that do not use this term for resulting in general, it is at least likely to belong together with the complex of place-value computation (cf. above, p. 6).

The third multiplication is "repeating". We have encountered it above, as one of the possible ways to express the circular perimeter in terms of the diameter (p. 2). The main term is *eṣēpum* / t a b ,[42] with n i m as an occasional alternative. When occurring without a specification "to *n*", its meaning is doubling. Except in a few instances in the series texts, *n* is always smaller than 10, and the term

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[40] Two of the problem texts from Ur, UET 5,864 and UET 5,858, have the phrase *a* a . r á *b* ù . u b . r á , "*a* steps of *b*, when you go". AO 8862 ("group 1", cf. above) uses *a* a . r á *b* repeatedly where we would expect a "holding" or (once) an ordinary halving (for both operations, see below). The two theme texts from "group 2A" (YBC 4662, 4663) apply it a few times, in the phrases *a* a . r á *b i-ši*, "*a* steps of *b* raise", or *a* a . r á *b* UR.UR.A, "*a* steps of *b* make hold". The atypical Susa text TMS XXVI [Muroi 2001: 229*f*] has the purely numerical sequences 26,40 a . r á 2 53,20 – 35 a . r á 35 20,25 – 1,20 a . r á 6 8 (misprinted in the edition) – and the sequential 1,20 a . r á 20 26,40 a . r á 2 53,20. Finally, some series texts (e.g., YBC 4668) couple it to "repeating" (see below), with phrases like a . r á 2 e . t a b , "(in) two steps repeated".

[41] That volume determination is the origin can be seen by the order of factors. When volumes are concerned, it is the base that is "raised" to the height. In all other situations, the order is determined by the textual structure, the number which has just been found being "raised" to the other factor.

[42] The basic meaning of t a b being "to be / make double, to clutch, to clasp to" [Thomsen 1984: 318], the logogram is obviously not very adequate but a secondary choice, derived from one of the meanings of the Akkadian term.

always refers to a concrete *n*-doubling of the tangible entity concerned, not to a mere numerical multiplication.

In the Susa corpus (TMS VII, VIII), syllabic forms of *alākum*, "to go" (until *n*), occur both as equivalents of *eṣēpum* and when an "appending" is to be repeated. This reveals an underlying conceptual connection between the operations of "steps" and "repetition", as also confirmed by the Ur occurrences of the phrase *a* a . r á *b* ù . u b . r á , "*a* steps of *b*, when you go" and of the use of certain series texts of the phrase a . r á *n* e . t a b , "(in) *n* steps repeated" – cf. note 40.

Rectangularization and squaring

A term which is traditionally also translated as "multiplication" is *šutakūlum*, with a number of logographic equivalents. Actually, it stands for the construction of a rectangle with sides *a* and *b*. As a rule, the calculation of the area is understood to be implied in the process, but if the rectangle is already there, its area is found by "raising", showing that *šutakūlum* cannot be a mere area determination.

The verb is the causative-reciprocative form ("make ... each other" or "make ... together"), either of *akālum*, "to eat" (the guess of Neugebauer), or of *kullum*, "to hold")(that of Thureau-Dangin). Since that which has been caused to "eat" / "hold" can either be referred to by the relative phrase *ša tuštakil* or by the noun *takīltum*, which can only be derived from *kullum*, there is now no doubt that Thureau-Dangin was right;[43] moreover, since the double object (the two segments *a* and *b*" that are "caused to hold") are sometimes connected by the preposition *itti*, "together with", the meaning must be "make *a* and *b* hold together". Even though there is no reason to assume semantic continuity (nor to exclude it), the idea is thus the same as in Greek geometry: even here, a rectangle is "contained" or "held" (περιέχω) by two sides (*Elements* II, def. 1 [ed. Heiberg 1883: I, 118]).

[43] An apparent counter-argument is the use of the logogram g u $_7$. g u $_7$, "eat-eat". However, Sumerian reduplication did not correspond to Akkadian causative-reciprocative, and the logogram is thus clearly a secondary construction, formed from the Akkadian (as are the other reduplicated logograms, cf. below), and such a secondary construction could easily be inspired by the quasi-coincidence of the corresponding forms of *kullum* (*šutakūlum* or possibly *šutakullum*) respectively *akalum* (*šutākulum*) – such puns or rebus-writings had been the fundament for the whole development of cuneiform writing from the purely logographic-pictographic script of the fourth millennium.

The construction of a square with side *a* may be described by the same term (either "making *a* and *a* hold" or just "making *a* hold"); but it may also be spoken of with the equally causative-reciprocative *šutamḫurum*, "to make (*a*) confront itself", derived from *maḫarum*, "to confront (on a footing of equality)". To this corresponds the term *miṯḫartum* for the square configuration (literally something like "a situation characterized by the confrontation of equals"). Unexpectedly for us but in good agreement with the meaning of the word (which refers to the square frame, not to the area it contains), the numerical value of the *miṯḫartum* is the length of the side – a *miṯḫartum is* its side and *has* an area, while our square *has* a side and *is* an area.[44] If one side of a square has been found, the other side meeting it in a corner is referred to as its *meḫrum*, "counterpart".

Both *šutakūlum* and *šutamḫurum* have logographic equivalents, but most of these can stand for either of the Akkadian terms. g u$_7$. g u$_7$ was mentioned in note 43. Beyond that, there is UL.UL, almost certainly to be read d u$_7$. d u$_7$, properly "to butt each other" but according to backward syllabic references in relative phrases actually to be read *šutakūlum*; UR.UR – no certain explanation seems to be at hand, but cf. note 40; LAGAB, whose sign is a square frame, and which may be iconic, and LAGAB.LAGAB = NIGIN, which *may* combine the iconic aspect of LAGAB with the causative-reciprocative aspect of the reduplication. Because of the imperfect correspondence with the two Akkadian words, it may be better to see all these terms as *ideographic* (in the sense our mathematical symbols like "+" are ideographic) and not as genuine logograms.

The side of a square area (corresponding in modern but inadequate terms to the square root) is mostly expressed by the terms í b . s i$_8$, b a . s i$_8$ or (in Ešnunna) some unorthographic variant. In Ešnunna, the b a -variants are sometimes preferred, elsewhere (as a rule) these are reserved for cubic and quasi-cubic sides. This *may* but need not have to do with the different ways in which Ešnunna and the South inherited the Ur III tradition.

The full phrase of the inverse square tables is *A*. e *s* í b . s i$_8$ (cf. note 10), s i$_8$ meaning "to be equal". The final position í b . s i$_8$ shows this to be meant as a verb; the grammatical case of interrogatives shows that the interpretation in the Ur group was "close by *A*, *s* is equal", while all later groups that conserve the verb interpretation appear to have changed the reading into "*A* causes *s* to be equal".[45] A translation that renders both is "by *A*, *s* is equal".[46]

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[44] This is the real background to the nonsensical claim sometimes advanced, that the Babylonians did not distinguish a square from a square root.

[45] The Sumerian suffix . e may be terminative-locative as well as agentive.

However, not all text groups understand í b . s i$_8$ as a verb. “Group 7A” does, but most texts from “group 7B” understand it as a noun; probably this “equalside” is thought of as the kind of “thing” listed in the tables. “Group 1” is also uneven in its usage, while groups “2A” and “ii” opt for the verb. So does “group 3”, while “group 4”, the other Uruk group, and the single text from “group 5” that is relevant, opt for the noun. “Group 6” mostly asks and answers with the verb, but sometimes falls entirely outside the pattern, and states in syllabic Akkadian that *s imtaḫḫar*, “*s* confronts itself”, *s* t a . à m *imtaḫḫar*, “*s*, each, confronts itself”, or *s* í b . s i$_8$ *imtaḫḫar*, “*s*, as equalside, confronts itself”.[47]

Division and parts

As well known, division was no operation in sexagesimal place-value arithmetic. Division *problems* were of course well known (also in practical computation). If possible, the problem was solved via multiplication with the reciprocal; in practical computation this could always be done, since the technical constants that might turn up as divisors were always chosen so as to possess a simple sexagesimal reciprocal. In mathematical school texts, however, many division questions appear that cannot be solved in this way. Then the division question “what shall I posit to *P* which gives me *Q*?” is asked, and the answer stated immediately. Since the problems where it happens were invariably constructed backwards from known solutions, the answer would always exist and always be known to the author of the problem.

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[46] Quite unique in the corpus, YBC 6504 (an outlier in “group 1”) uses í b . s i$_8$ in two of four parallel passages for squaring, presumably for *šutamḫurum*, and d u$_7$. d u$_7$ in the others. The geometric text BM 15285 uses í b . s i$_8$ logographically for *miṯḫartum* meant as a geometric configuration.

[47] The phrase *a imtaḫḫar* is also found in BM 13901 #23, a problem that conspicuously leaves the canonical formulations of this long texts about squares and quotes a traditional riddle of the lay surveyors in their characteristic parlance – cf. [Høyrup 2002a: 222–226]. There is nothing jocular about the “group 6” texts; their use of the same phrase thus points to genuine vicinity to the same environment.

The question *kiyā imtaḫḫar*, “how much, each, stands against itself”, making even more clear that several sides are asked for, is found in the related texts CBS 43 and CBS 154+921 [ed. Robson 2000: 39*f*]. These texts are unprovenanced (because of too swift reading of Eleanor Robson’s publication I ascribed them to Nippur in [Høyrup 2002a: 354]). However, the writing of u š with a phonetic grammatical complement ^{ia}, “my”, suggests them to be early, probably contemporary with the Ešnunna texts; Robson tells me (personal communication) that they *may* be from Sippar – but they obviously do not belong to “group 6”.

This is the case in almost all text groups – the exceptions being the Ur group, where the formulation in UET 5,859 is somewhat different, and the series texts, where no prescriptions are present and the questions therefore do not arise. There is no reason to elaborate.

It is also well known but not much spoken about that the expression i g i *n* may as well refer to the reciprocal of *n* as to the *n*th part *of something*.

Originally, there was no difference. As shown by Piotr Steinkeller [1979: 187], some early tables of reciprocals (of Ur III date) make clear that they list not reciprocals in our sense but *n*th parts of 60 – an example is published in [Oelsner 2001: 56]. Obviously, that makes no difference in the numbers when written in a floating-point system, and in Old Babylonian times the reciprocal and the *n*th part were clearly distinct concepts.

There was no standard way to keep the two ideas clear of each other; all the more interesting is it that different texts, though using different verbal means, distinguish very clearly.

The basic term for the reciprocal of *n* is i g i *n* ĝ á l . b i , "[of 1], its i g i *n* ĝ á l ", whose meaning is enigmatic. i g i , originally the picture of an eye, is used as a logogram for *īnum*, "eye", for *amārum*, "to see", and for *pānum*, "face". The latter gave rise to an Old Babylonian folk etymology, the i g i of a number being either replaced by or glossed as *pāni*,[48] "in front of", namely "is placed (ĝ á l) in front of *n* in the table of reciprocals". However, the use of i g i for "part" goes back at least to the early 24th century, thus antedating the tables of reciprocal by 300 years or more. The only plausible explanation (whose central idea goes back to [Friberg 1978: 45]) is that the phrase means "*n* placed in eye", which would be a description of the proto-literate notation for fractions in the grain system [Damerow & Englund 1987: 136]. Since half a millennium without any fractions in the record separate the two notations, this can be nothing but a hypothesis.

Many tables of reciprocals carry the full phrase i g i *n* ĝ á l . b i , others abbreviate it into i g i *n* ĝ á l or i g i *n* . Of course, tables do not speak about the *n*th part of something; in order to see the distinction we must look at problem texts that refer both to reciprocals and to parts.

One possibility is ellipsis. The "group 3" texts Str 367, VAT 7532, VAT 7535, etc.) speak of the *n*th part of something (even of 1 if this number represents an unknown length in an argument by false position) by the phrase i g i *n* ĝ á l ; the number facing *n* in the table of reciprocals is simply igi *n*. In the tablet BM

[48] Replaced in Haddad 104, glossed in the "group 6" text BM 96957.

85210 (“group 6”) the same distinction is made, but supplemented by the use of different verbs: the reciprocal is “detached” (d u $_8$), as it always is; the *n*th part of *m*, however, is “torn out” (z i). BM 85194 (also “group 6”) uses the short form for both concepts, and distinguishes by the choice of verb alone.

Halves and halving

Old Babylonian Akkadian mathematics distinguishes two “halves”. One belongs to the same general class as $^1/_3$, $^1/_4$, etc. This half may be a number (30´) or the half of something. It can be written syllabically (*mišlum*); as 30´; with the sign BAR (+); or with the Sumerogram š u . r i . a .

The other is a “natural half”, invariably *of something*. It is mostly spoken of as *bāmtum* (in general language “half-share”, one of two opposite mountain-ridge slopes or body parts),[49] but in Db$_2$-146 (“group 6B”) it appears as *muttatum* (generally “half-pack” etc.). It is used in situations where no fraction but the half would do – the radius as part of the diameter, the half of the base of a triangle serving in area computation, etc. It has no proper logogram, but the strongly logographic text YBC 6504 (the “outlier text” from “group 1”) use š u . r i . a , while groups “3” and “6” as well as the Susa texts sometimes or always use BAR.

The operation by which a natural half is produced is “to break” (*ḫepûm* / g a z).[50] *ḫepûm* as well as g a z have the general meaning “to smash”, “to destroy”, “to break (into any number of parts)”. This thus presents us with a rare case of clearcut separation of technical and general-language meaning – quite different from what we saw in the case of removal-subtractions.

Kassite survival

We have very little evidence for any kind of mathematics from the Kassite period (c. 1600 to c. 1200), nor indeed indirect evidence of the kind we have for the Kassite unfolding (after late Old Babylonian inception) of systematization of fields like incantation, medicine and extispicy. It seems that the scribal families that took care of the conservation of scribal scholarship did not care for the survival of mathematical sophistication.

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[49] The hypothetical **bûm* of [MCT, 161] (cf. [CAD II, 297] and [AHw I, 116]) ascribed to mathematical texts is constructed from *ba-a-šu* and similar forms, which almost certainly correspond to a contracted form *bāššu* (<**bāmat-šu*) of *bāmtum*+possessive suffix *-šu*.

[50] I have noticed only one exception to this rule: the “group 7B” text IM 43993, which uses *letûm*.

One text, AO 17264, looks as an exception to this rule (the dating is made on the basis of palaeography; the dealer claimed the tablet to be from Uruk). It is a procedure text about a very intricate problem, the partition of a trapezoidal field between six "brothers" into strips that are pairwise equal. Actually, the problem is too intricate for its author, and the solution is no mathematical solution. Lis Brack-Bernsen and Olaf Schmidt conclude [1990: 38] after analyzing the text that the problem

> is beyond the capability of Babylonian mathematicians, and it looks as if they have given up in despair in their attempt at solving this problem and just given some meaningless computations that lead to a correct result.[51]

But this is not our primary concern here. More interesting are the problem format and the terminology. The statement first tells the object, and asks an explicit question e n . n a (much faster to write than e n . n a m). The prescription starts z a . e k ì . d a . z u . d è , "you, by your proceeding" (in the "southern" spelling of groups "2" and "3"), and ends *kīam nēpešum* – a formula known only from Old Babylonian texts belonging to groups "6", "7" and "8". The plane "equalside" is a noun and "comes up" – *ba-se-e-šu šu-li-ma* – the phrase as well as the unorthographic spelling points to "group 7B". Results are followed by I.DÙ; both Neugebauer and Thureau-Dangin understand this as ^{i}d ù ~ibanni, "it produces" (literally "it builds"), which would be an absolute innovation;the complement *i* suggests that this may indeed have been the scribe's own understanding. But the spelling i g i . d ù instead of i g i . d u $_8$ in IM 55357 suggests that the historical root of the innovation is a reinterpretation of the unorthographic Ešnunna spelling of "seeing" – another (scholar's) folk etymology".

Unorthographic spelling also seems to explain í b . TUG, used twice after a removal: As proposed by Thureau-Dangin, the word is likely to stand for *šapiltum*, which would regularly be written í b . t a g $_4$.

Accumulation is UL.GAR, as in groups "6A" and "8A", while squaring is UR.KA – apparently a cross-breed between UR.UR (YBC 4662–63, "group 2") and KA+GAR (TMS XXVI, "group 8C"). LAGAB, elsewhere used as a logogram for squaring and rectangularization, is used instead to tell the equality of shares (probably intended as s i $_8$). "Breaking" is treated as in "group 7A", mentioning neither that it is "into two" (as in "group 4") nor the resulting natural half (as habitual elsewhere).

[51] It is indeed not too difficult to construct a statement with adequate parameters and a known solution from the table of squares.

Apart from the spelling of the introductory formula, the features are thus definitely "northern", but vacillating between groups 6A, 7A+B and 8A+C, with preponderance for the links to "group 7". If the tablet is really from Uruk, the southern tradition must have been so brutally interrupted that sophisticated mathematics had to be imported anew during the Kassite period. Since dealers are not necessarily to be relied upon, the text may also represent the left-overs of the northern tradition without being strictly descended from any of the groups which accident has allowed us to discover.

Fifth-century scholar-scribes

We know that Assurbanipal claimed in the mid-seventh century to be able to perform multiplications (a . r á) and to "detach" reciprocals (*u-pa-ṭar* i . g i),[52] which shows survival of the basic terms of sexagesimal place-value computation within the environment where the future king had received his scribal training. But we have to wait another couple of centuries before two texts containing mathematical problems turn up [Friberg, Hunger & al-Rawi 1990; Friberg 1997]. As can be read in a colophon, these texts belonged to a scholar-scribe from fifth-century (thus Achaemenid) Uruk.[53] At least the text carrying a colophon was copied from a wax tablet, probably by the owner.

The problem format in these texts is rudimentary. They start by presenting the situation, probably in grammatically neutral form (sometimes certainly, sometimes the use of logograms may hide an intended first person singular), and then mostly specify the question with e n (an even more radical abbreviation of e n . n a m). The prescription is formulated in the second person singular and either devoid of opening formulae or introduced m u n u z u . *ti*, "since you do not know".[54] Sometimes, the prescription is formulated in general terms

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[52] [Ungnad 1917: 41*f*], revised interpretation. Later quotations of the text, such as [Fincke 2003: 111], tend to understand its mathematics less well than Ungnad did.

[53] Namely to Samaš-iddina, "son of Nādinu, descendant of Šangi-Ninurta, exorcist from Uruk" [Friberg, Hunger & al-Rawi 1990: 545], dating [Robson 2008: 227–237].

In order to avoid wrong connotations to Catholicism or modern occultism (mathematics is not the only field where wrong connotations turn up!), it might be better to translate the profession of the forefather of the scribal family as "ritual specialist".

[54] n u z u and the syllabic equivalent *lā tidū* also appear in Old Babylonian groups "1", "3" and "7A", but not as opening formulae for the prescription. Since absence of knowledge is inherent in the problem situation and n u z u its simplest expression, reinvention of the same formula is far from excluded.

and not as a specific numerical paradigmatic example. Often, the calculation is made in two ways, "if (*šumma*) 5´ is your cubit" and "if 1 is your cubit", corresponding of the choice of the n i n d a n (12 cubits. c. 6 m) respectively the cubit as the basic unit for the sexagesimal calculations. In the Old babylonian period, the cubit was used as the basic unit for vertical distances only. Could it be that the corresponding metrological table had survived in the scholarly environment but its particular use had been forgotten?

Both texts are concerned with new area metrologies, one based on "broad lines" (cf. note 36 and preceding lines), the other on the standard expectation concerning the grain needed for sowing and for feeding the plough oxen. Both correspond to the habits of genuine surveyors.

Some of the problems are "algebraic" in nature – not derived, however, from the fully developed Old Babylonian discipline but from the simple riddles that had once inspired it.

Part of the terminology for operations has Old Babylonian antecedents. g̃ a r . g̃ a r and d a ḫ , respectively "to accumulate" and "to append", are both used as traditionally (always written logographically). Subtraction, however, is made by "lifting", that is, n i m , a term that in Old Babylonian texts had been used occasionally as a logogram for *našûm*, "to raise" – cf. above, p. 4. The transmission in Sumerian must thus have been partially interrupted, and a new translation of Akkadian (or, by now, Aramaic) terms into Sumerian must have taken place. í l , the other logogram for "raising", has conserved its meaning, and the syllabic *našûm* may also be encountered. Constructing a square is *maḫārum* (syllabic, but not *šutamḫurum*). Often, multiplication is a . r á *n* r á , "steps *n* go", similar both to the Ur expression *a* a . r á *b* ù . u b . r á (above, p. 25), and to that of various series texts, a . r á *n* e . t a b , "(in) *n* steps repeated". The "equalside" is UR.A, but in order to find numerically theequalside of *A*, the phrase *A*. e à m t i ^{qe} (t i ^{qe} = *leqe*, "take") may be used, with the alternative i b . s a , unorthographic for í b . s á = í b . s i $_8$. Friberg proposes [Friberg, Hunger & al-Rawi 1990: 509] that the former formulation may be an abbreviated reference to a formula used in a few Old Babylonian tables of inverse squares, *A*. e *s* . à m íb.si$_8$.[55]

Results are mostly marked by a preceding enclitic -*ma*, but final results often by i g i -*mar* or *tammar*, "you will see". The general rules may also refer to an intermediate result (which because of the abstract formulation cannot be identified numerically) as *šá ana* i g i -*ka* e $_{11}$ -*a*, "what for your eye comes up"

[55] Friberg transcribes à m as a a n , but that makes no difference.

[Friberg, Hunger & al-Rawi 1990: 536] – a combination of the two ways results were announced in group "7A", whose closeness to the riddle tradition ("If somebody ...") we noticed.

All in all, these texts, like the Kassite AO 17264, confirm that the "southern" post-Hammurabi traditions as represented by groups "1" through "4" had no conspicuous influence in what little problem culture survived the Old Babylonian collapse. Transmission within scholarly (that is, Sumerian-trained) and less scholarly but still schooled practitioner's environments as well as within orally based milieus of lay practitioners probably participated in the process, but it is difficult to extricate their respective roles.

The Seleucid texts

Three Seleucid problem texts are known: VAT 7848, AO 6484 and BM 34568. A colophon in AO 6484 states that it was written by the astrologer-priest Anu-aba-utēr, member of a scribal family descending from the astrologer-priest Sîn-leqe-unninnī from Uruk. Anu-aba-utēr was active in the early second century [Hunger 1968: 40 #92 and *passim*]. The colophons of the other two texts are destroyed, but they appear to come from the same scholar-scribes' environment and to be roughly contemporary.

The problem format is rudimentary. The statement *may* start by stating the object, but mostly only describes the situation, apparently in grammatically neutral form; there is no closing formula, and no explicit question except when it is not clear what is meant. In BM 34568, the prescription starts m u n u z u^{ú} (the phonetic complement indicating an Akkadian pronunciation *assum la tidû*, "since you do not know"), and it can be seen to be meant to be in the second person singular. In the other two texts, there is no opening formula, and the prescription appears to be grammatically neutral.

As concerns the operations, "accumulation" has become g̃ a r in BM 34568 but remains g̃ a r . g̃ a r in the other two. The identity-conserving addition has become *ṭepûm*, mostly written logographically t a b – which, we remember, was used for "repetition" in the Old Babylonian texts. Just as in the case of n i m in the fifth-century texts, we have evidence of a re-Sumerianization of the vernacular language and thus of interruption of the tradition at the scholarly level.

Similar evidence comes from the terms for subtraction. Beyond n i m , which is still used "lifting up" from the reckoning board), removal may be designated l a l , which in Old Babylonian times had been used for comparison "the other way round" (above, p. 23).

Multiplication is *a* GAM *b* or *a* GAM *b* r á , where the easily written repetition sign GAM (in the three-stroke variant) is obviously used as a logogram for a . r á .

All variants of í b . s i$_8$ (the "equalside") have disappeared, and so has the enigmatic fifth-century use of à m in the same function. Instead, these texts ask for the square root of *A* in a purely arithmetical phrase, "how many steps of what shall I go so that *A*?".[56]

Several problem types from the two texts AO 6770 and BM 34568 that have no known antecedents in Mesopotamia turn up in Demotic papyri from the same epoch [Høyrup 2002b]. The scholar-scribes from Uruk never went there, they had nothing to do with the Assyrian, Achaemenid and Macedonian armies and tax collectors that had been customary visitors of Egypt since centuries. Even the contents of the problems thus confirms that the scholar-scribes adopted much of their mathematics from practitioners who did go around the world

The mathematical terminology of astronomy

– yet certainly not all of it. These ritual specialists and "scribes of [the astrological omen series] *Enūma Anu Enlil*" were also those (or some of them were) who produced mathematical astronomy. The tendency toward arithmetization which we see in the transformed question for the square root is likely to have been inspired by their extensive numerical work; even though the many-place tables of reciprocal produced in Seleucid times probably had no direct function in astronomical calculation, they may also be an abstract spin-off from the same numerical practice.

Planetary tables in themselves contain no terminology for the mathematical operations involved in their production. However, another astronomical genre does: the procedure texts.

One of these – BM 42282+42294, a probably Achaemenid text from Babylon or Borsippa – explains the "goal-year method". It contains no problems, so we should not look for any problem format. What we find is a terminology for additive and subtractive operations.

Certain Old Babylonian terms that have disappeared from the Late Babylonian problem texts survive here: *kamārum* (written phonetically) as well as g̃ a r . g̃ a r for "accumulation", and z i for subtraction (the latter probably meant as "lifting" since operations on the "hand" are explicitly spoken about; even g̃ a r . g̃ a r could be meant as "positing" on the reckoning board). But the

[56] *mu-nu-ú* GAM *mi-ni-i lu*-r á *-ma lu A*. The genitive *mi-ni-i* removes any possible doubt that GAM really stands for a . r á , "steps of".

identity-conserving addition (whether thought of as *waṣābum* or *ṭepûm*) has become t a b , as in the Seleucid texts (above, p. 33).

The "handbook" MUL.APIN [ed. Hunger & Pingree 1989: 101], known among other places from Assurbanipal's library and not necessarily much older than the initial seventh century, shows us that "raising" (written í l) was still in use, and that the outcome of a calculation might be "seen" (*tammar*). But this was written when mathematical astronomy was at most in its most primitive beginnings. Half a millennium or more separates it from our Seleucid texts.

Texts referred to, with location of publication

AO 6484: MKT I, III.
AO 6770: MKT II.
AO 8862: MKT I.
AO 17264: MKT I.
BM 13901: Thureau-Dangin 1936, MKT III.
BM 15285: MKT I.
BM 34568: MKT III.
BM 42282+42294: Brack-Bernsen & Hunger 2008.
BM 85194: MKT I.
BM 85196: MKT II.
BM 85200 + VAT 6599: MKT I.
BM 85210: MKT I, III
BM 96957: Robson 1996.
CBS 43: Robson 2000.
CBS 11318: Neugebauer & Sachs 1984.
CBS 154+921: Robson 2000.
Db_2-146: Baqir 1962.
Haddad 104: al-Rawi & Roaf 1984.
IM 43993. Unpublished, courtesy of Jöran Friberg and Farouk al-Rawi.
IM 52301: Baqir 1950b.
IM 52916+52685+52304: Goetze 1951.
IM 55357: Baqir 1950a.
IM 121613. Unpublished, courtesy of Jöran Friberg and Farouk al-Rawi.
IM 52916+52685+52304: Goetze 1951.
Ni 18: Proust 2008.
Plimpton 322: MCT
Str 367: MKT I.
TMS, all texts: TMS.
UET 5,858: Friberg 2000.
UET 5,859: Friberg 2000.
UET 5,864: Friberg 858.
UM 29-15-192: Proust 2008: 180–183.
VAT 7528: MKT I.
VAT 7532: MKT I, III.
VAT 7535: MKT I.
VAT 7848: MCT.
VAT 8521: MKT I.
VAT 8523: MKT I, III.
YBC 4608: MCT.
YBC 4662: MCT.
YBC 4663: MCT.
YBC 4668: MCT.
YBC 4669: MKT I, III.
YBC 4673:MKT I, II, III.
YBC 4675: MCT.
YBC 4698: MKT III.
YBC 4714: MKT I.
YBC 6295: MCT.
YBC 6504: MKT III.
YBC 6967: MCT.
YBC 7289: MCT.
YBC 7290: MCT.
YBC 7302: MCT.
YBC 8600: MCT.
YBC 8633: MCT.
YBC 11126: MCT.

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