Sequences and Series
in
Old Babylonian Mathematics

Duncan J. Melville
St. Lawrence University

Old Babylonian mathematics contains many interesting word problems on a wide variety of more or less practical topics. Among them are a number involving various (finite) sequences and series, usually based on arithmetic, geometric or harmonic (i.e., reciprocal) progressions. For example, two problems studied below may be stated in modern dress as:

> 10 brothers share $100 (in descending arithmetic progression). The share of the eighth brother is $6. How much was the difference between shares?
>
> I had a measuring rod. Each time I measured, 1 cm fell off. By the time it was all gone, I had measured 120 cm. How long was the rod originally?

Mesopotamian scribes had a propensity for ordering their data and it is natural that we find ordered mathematical data. From there, it is a short step to constructing problems based on regular changes in some variable. In this paper, we indicate some of the types of ordered data, and arithmetic problems and word problems based on progressions to be found in the corpus of Old Babylonian mathematics. The paper does not attempt to provide an exhaustive survey of these types of problems, but the examples chosen do show some of the range of techniques used to solve such problems, especially the more complicated inverse ones.

Ordered Data

It is well-known that the scribes of Mesopotamia had a passion for organizing data into standard forms or lists, and that these lists had a pedagogical function. Already in the archaic period of the earliest writing (Uruk IV-III, 3300-3000 BC) there are many examples of tablets with extracts of standard lexical lists such as the profession list known as Lu-A[1]. Over the course of the third millennium there was a great expansion and codification of these lexical lists and a great many examples are known numerous sites.

[1] A large number of these archaic lexical lists were published by Englund and Nissen [2]; they can now be accessed via the CDLI website http://cdli.ucla.edu/. A related project is the Digital Corpus of Cuneiform Lexical Texts (DCCLT) under the direction of Niek Veldhuis, which aims to present the entire corpus of lexical texts (http://cdli.ucla.edu/dcclt/dcclt.html).

By the time of the Old Babylonian period (2000 – 1600 BC), the best-documented period of Mesopotamian mathematics, these lists had an additional educational function, for students learned to write Sumerian, and developed their Sumerian vocabulary through copying such lists, as Sumerian was no longer a living, spoken language. As students advanced, they graduated to more complicated texts. Recently, the outlines of the scribal syllabus have been unraveled by Niek Veldhuis [24, 25], while Eleanor Robson has helped to fit the study of mathematics into place, at least in the case of schooling at Nippur [17, 18]. According to this scheme, the first sustained exposure students would have with mathematical material would be copying out so-called metrological lists. Such lists recorded quantities of capacity, weight, area and length from very small units to the very large using standard metrological units and notation. For example, the full capacity list started with ⅓ sìla (about ⅓ liter) and ended with 1,00,00 gur (approximately 65 million liters) [3]. Changes in step-size as the quantities increased meant that the lists, although large, were not of unmanageable length. These metrological lists are commemorated in the comment: 'I want to write tablets: the tablet (of capacities) from 1 gur of grain to 600 gur; the tablet (of weights) from 1 gín to 10 mana of silver' from one of the *eduba* or school-house texts [1]. An extract from such a list is given in Table 1.[2]

18	gur
19	gur
20	gur
30	gur
40	gur
50	gur
1,0	gur
1,10	gur
1,20	gur
1,30	gur
1,40	gur
1,50	gur
2,0	gur
3,0	gur
4,0	gur
5,0	gur
6,0	gur

Table 1: Extract from a metrological list

A little more complicated than metrological lists were metrological tables.[3] These tables were based on the great innovation of the late third millennium that made Old Babylonian mathematics possible – the development of the abstract sexagesimal numeration system.[4] Whereas a student preparing a metrological list merely wrote out quantities in correct units, a student compiling a metrological table recorded first the quantities in proper metrological units in one column and then the

[2] The extract is taken from Column III of the obverse of CBM 10990+19815+19757 (BE 20/1, 29), a metrological tablet of which three fragments exist, adding up to only a portion of the original. Published by Hilprecht in [5]. The original tablet had six columns on the obverse and six on the reverse. For the multiplies of gur given in the extract, the cuneiform notation is similar to the abstract sexagesimal system (base 60) and has been translated as such using the convention of commas to separate sexagesimal places.

[3] This distinction between 'lists' and 'tables' is the standard usage. More recently, Robson [19] has argued for a more precise and restricted usage of the term 'table'. In Robson's typology, tables should have more structured formatting with both horizontal and vertical calculation.

[4] The sexagesimal system is a place-value system with a base of 60. There are several conventions for transliterating sexagesimal notation. Here, we follow the Neugebauer convention of separating sexagesimal places with a comma, and indicating absolute size where necessary with a semi-colon. Thus, 5,30 denotes 330 but 5;30 denotes 5 ½.

corresponding equivalents in terms of sexagesimal multiples and fractions of a base unit in another column. In most Old Babylonian mathematical problems, the statement of the problem and the solution were given in the appropriate units, but intermediate computations were carried out in the abstract system based on an (often only implied) unit. Thus, it was important for scribes to practice converting between metrological values and the sexagesimal system. Table 2 gives an extract from an Old Babylonian metrological table (BE 20/1, 32) recording multiples of še and metrological fractions of a gín and converting them into sexagesimal fractions of a gín. The metrological relationship is that 1 gín contains 180 (or 3,0 in the sexagesimal system) še.

The core of Old Babylonian mathematical practice, greatly facilitated by the new place-value system, was multiplication and the inverse operation of division, effected in Mesopotamia by multiplying by the reciprocal. Hence, it was important for students to practice multiplication facts, and, as usual, the practice was organized into tables, in this case multiplication and reciprocal tables, of which some 300 examples survive. Additionally, we have several examples of tables of squares, square roots, exponentials and associated organized mathematical data.[5] Two examples of these are given in Tables 4 and 5, one a short series of square roots (Ist Ni 2739, published in *MKT* I) and the other a list of exponentials (Ist. O 3816, also originally published by Neugebauer in *MKT* I).

20	še	0;6,40
21	še	0;7
22	še	0;7,20
23	še	0;7,40
24	še	0;8
25	še	0;8,20
26	še	0;8,40
27	še	0;9
28	še	0;9,20
29	še	0;9,40
1/6	gín	0;10
1/6	gín še 10	0;13,20
¼	gín	0;15
¼	gín še 5	0;16,40
⅓	gín	0;20
½	gín	0;30
⅔	gín	0;40
5/6	gín	0;50
1	gín	1

Table 2: Extract from a metrological table

1	e	1
1,2,1	e	1,1
1,2,3,2,1	e	1,1,1
1,2,3,4,3,2,1	e	1,1,1,1

Table 3: A table of square (roots)

[5] Most of these tables have been minimally published by Neugebauer in MKT [14] and Neugebauer and Sachs in MCT [15]. Since the latter work appeared, a few more such texts have been published.

14,3,45	a.rá	3,45
52,44,3,45	a.rá	3,45
3,17,45,14,3,45	a.rá	3,45
12,21,34,37,44,3,45	a.rá	3,45
46,20,54,51,30,14,3,45	a.rá	3,45
2,53,48,25,43,8,22,44,3,45	a.rá	3,45

Table 4: An exponential sequence

Multiplication tables are organized in a number of different ways. Neugebauer and Sachs identified three common types and four more less-common variants [15: 20] depending on the precise terminology used. This detailed typology has been recharacterized by Robson as representing 'verbose' and 'terse' texts. Further, Robson has suggested that verbose tables were written for initial practice, while terse tables, especially combined multiplication tables with several different principal numbers, were composed for later review [19]. Ignoring these minor typological differences, a typical multiplication table would have a format such as in Table 3 (NBC 7344). Each multiplication table has a principal number (for Table 3 the principal number is 5) and records multiples of that principal number from 1 to 19, and then 20, 30, 40, and 50 times. In almost all cases, the principal number is a regular number appearing in the standard table of reciprocals.[6] The close connection of the multiplication tables to reciprocals is emphasized by the large combined tables that begin with a reciprocal table and continue with a series of multiplication tables (See *MKT* and *MCT* for examples).

5	a.rá 1	5	a.rá	12	1
a.rá	2	10	a.rá	13	1,5
a.rá	3	15	a.rá	14	1,10
a.rá	4	20	a.rá	15	1,15
a.rá	5	25	a.rá	16	1,20
a.rá	6	30	a.rá	17	1,25
a.rá	7	35	a.rá	18	1,30
a.rá	8	40	a.rá	19	1,35
a.rá	9	45	a.rá	20	1,40
a.rá	10	50	a.rá	30	2,30
a.rá	11	55	a.rá	40	3,20
			a.rá	50	4,10

Table 5: A multiplication table

[6] A sexagesimal number is called 'regular' if its only prime divisors are 2, 3, and 5, the divisors of 60. Regular numbers have finite sexagesimal reciprocals. A rare example of a multiplication table with principal number not regular was published recently by Nemet-Nejat; it has principal number 13 [12].

The standard reciprocal table began, ‘1 its ⅔ is 40; its ½ is 30’, and continued with lines of the form ‘igi *n* gál-bi $\bar{n}$ ’ , ‘the reciprocal of n is $\bar{n}$ ’ for regular numbers *n* from 1 to 1,21 (= 81). The formatting was thus similar to that of the multiplication tables. The reciprocal pairs, though not with the standard formatting, are given in Table 6.

2	30	16	3,45	45	1,20
3	20	18	3,20	48	1,15
4	15	20	3	50	1,12
5	12	24	2,30	54	1,6,40
6	10	25	2,24	1	1
8	7,30	27	2,13,20	1,4	56,15
9	6,40	30	2	1,12	50
10	6	32	1,52,30	1,15	48
12	5	36	1,40	1,20	45
15	4	40	1,30	1,21	44,26,40

Table 6: The standard reciprocals

Since reciprocals were so important in Old Babylonian mathematics, students needed practice finding reciprocals of larger regular numbers, and there was a standard procedure, dubbed ‘The Technique’ by A. Sachs, for finding these reciprocals [20]. A variant procedure was described by Friberg [4]. Several tablets are known that contain sequences of reciprocal pairs, presumably generated to provide problems for students. These sequences of reciprocals often began with the favorite pair 2,5 ~ 28,48 and continued by a process of doubling and halving. For example, the tablet BM 80150 (see *MKT* I) begins with a reciprocal table, continues with 18 multiplication tables, and ends with a sequence of reciprocal pairs. The end of this sequence is very broken, the rest is given in Table 7. The early part of the sequence is also found on CBM 10201 [5: 25]; pairs from further on in the sequence appear, sometimes incorrectly, on other tablets. For example, YBC 10802 has the pair 2,22,13,20 ~ 25,12,42 [which should be 25,18,45], and MLC 651 has the pair 1,20,54,31,6,40 ~ 44,29,37,50,15,20 [instead of 44,29,40,39,50,37,30] (see [20] for a discussion of these and other examples).

2,5	28,48	igi-bi
4,10	14,24	igi-bi
8,20	7,12	igi-bi
16,40	3,36	igi-bi
33,20	1,48	igi-bi
1,6,40	54	igi-bi
2,13,20	27	igi-bi
4,26,40	13,30	igi-bi
8,53,20	6,45	igi-bi
17,46,40	3,22,30	igi-bi

Table 7: Part of a reciprocal sequence

Reciprocal Series

In all of the examples given above, the purpose of the iterative arithmetic computation is to create organized lists or tables of mathematical data. There is no calculation beyond the generation of the list. However, an intriguing collection of texts, now part of the Yale Babylonian Collection, do go further. These tablets were originally published by Neugebauer and Sachs in 1945 [15] and have now been re-published with photographs of the tablets and some additional examples by Nemet-Nejat [13]. They are all quite small (about 8cm on a side), squarish, and of the type that would typically contain some workings or computations for a single exercise.[7] In the case of this collection, the tablets contain tables laying out data involving a sequence of reciprocals and their sum.

A representative example is YBC 7234, the contents of which are given in Table 8.

			2,51,30
1	1	1,10	1,10
2	30	35	1,10
3	20	23,20	1,10
4	15	17,30	1,10
5	12	14	1,10
6	10	11,40	1,10

Table 8: A reciprocal series

Reading from left to right, the first column gives a sequence of numbers, $n = 1,2,3,4,5,6$. The second column contains the reciprocals, or perhaps more accurately inverses, of these numbers $\overline{n} = 1,30,20,15,12,10$. The third column multiplies these entries by a common constant, $c = 1,10$, and the fourth column multiplies the first and third to act as a check, $\overline{n}cn = c$. The number at the top of the fourth column, here 2,51,30 is the sum of the entries in the third column. That is, in modern notation, what is being computed is the series $\sum_{n=1}^{6} c\overline{n}$.

Among the remaining texts, YBC 7354, YBC 7355, and YBC 11127 are the same except for having $c = 30, 40$ and 2 respectively. YBC 7358 has $c = 45$, but n only goes up to 5. YBC 7235 has $n = 1,3,4$ and $c = 40$, but also includes some additional columns and computation. Rather more intriguing are the pair of texts YBC 7353 and YBC 11125. Although the layout and computations are similar, in both of these cases, the numbers starting the problem are the sequence of popular irregular numbers $n = 7,11,13,14$. Since these numbers are irregular, they do not have finite sexagesimal reciprocals. Instead, their inverses are taken with respect to the least common multiple $16,41 = 7 \times 11 \times 13$. So

[7] The relevant tablets are YBC 7234, YBC 7235, YBC 7353, YBC 7354, YBC 7355, YBC 7358, YBC 11125 and YBC 11127.

the inverse of 7 is 2,23, and so on. The example of YBC 7353 is given in Table 9. The constant is $c = 30$ and the total is the sum of the entries in the third column. In YBC 11125, the only difference is that there $c = 40$.

			3,11,15
7	2,23	1,11,30	8,20,30
11	1,31	45,30	8,20,30
13	1,17	38,30	8,20,30
14	1,11,30	35,45	8,20,30

Table 9: Irregular numbers and inverses

These texts contain only numbers, there is no explanation as to why these series are being computed, and a natural question is to ask what the exercises were for. It has been suggested, most forcibly by Friberg [3: 547], but also by Neugebauer and Sachs [15: 18-19], that these tables functioned as aids for solving 'price-equivalency' problems. The idea is that the computations calculate the total cost of equal amounts a number of items of different prices (prices in Mesopotamia were usually quoted as 'quantity per unit of silver' (pounds per dollar) rather than our 'cost per unit of quantity' (dollars per pound)). However, this argument is based on one poorly-understood text, VAT 7530, and it seems more likely that the artificial word-problems were constructed to utilize the prior calculations. The recent publication by Robson of another similar text from Nippur, N 3914, in [16] makes the price-equivalency hypothesis even less plausible. In N 3914, the sum runs from $n = 1$ to $n = 10$. It is difficult to believe anyone would want to purchase equal quantities of a collection of goods with market rates that just happened to run from 1 to 10. Further, since $n = 7$ is irregular, the inverses are all taken with respect to 7. The table is reproduced in Table 10. In view especially of this additional new example, these tables should be seen as pure exercises in calculation with the sexagesimal system with the emphasis on computing with inverses.

		3,25,1,40	
1	7	1,10	1,10
2	3,30	35	1,10
3	2,20	23,20	1,10
4	1,45	17,30	1,10
5	1,24	14	1,10
6	1,10	11,40	1,10
7	1	10	1,10
8	52,30	8,45	1,10
9	46,40	7,46,40	1,10
10	42	7	1,10

Table 10: Inverse computations from Nippur

Series in Word Problems

While it is quite possible that the texts described above were pure exercises in computation, various sorts of series do turn up in the corpus of word problems. There are problems involving arithmetic, geometric and more complicated types of sequences on a variety of topics. Some of the problems are direct, some are inverse problems (given the sum of a sequence and some conditions, find the terms), some are spectacularly artificial, some are quite subtle, and, unfortunately, many are badly broken. There are series problems on interest calculations and problems of inheritance involving division of silver or land between a number of claimants; there are problems involving workers carrying bricks and building ramps; there are questions on sizes of fields, and there are 'broken reed' problems. Below we give some representative examples of the genre.

Division of Silver

A classic series problem involves the division of an inheritance of silver between a number of brothers when the shares of the brothers follow a certain pattern. While there is some evidence for a basis in reality of the ideas of division of property in varying shares, the details of the problems are quite artificial and can safely be viewed as mathematical exercises. The entire collection of inheritance problems was surveyed by Muroi [10]. The examples given below indicate some of the variety of these problems. As may be expected, both of these are inverse problems. The total amount of silver to be divided between the brothers is known, and the problem is to find the share of each brother. The terminology is often cryptic and the problems are underdetermined at first sight. That is, crucial information, in particular the type of progression, is often not clearly stated.

The first example is taken from YBC 9856, originally published by Neugebauer and Sachs in *MCT* [15:99-100]. The tablet contains two problems, and both of them present serious difficulties in interpretation. The first exercise is to do with some kind of work assignment, the second is to do with division of silver. The translation given here is similar to that of Neugebauer and Sachs, despite the criticisms of Muroi [10].

> 1 mana of silver. 5 brothers. As much as
> the difference between two brothers is the share of the youngest.
> Let brother exceed brother.

The meaning is that 5 brothers share 1 mana or 1,0 = 60 gín of silver with 'brother exceeding brother' by a constant amount, i.e., in arithmetic progression. The share of the youngest brother (that is, the smallest share) is equal to the difference between successive shares. The text does not record how the problem was to be solved, but it does give the (correct) solution in the next line as a laconic set of numbers:

> 1 4 2 8 3 12 4 16 5 20.

It is entirely possible that the problem was intended to be solved by a method of 'false position', since many other problems used this technique, and this approach would yield the solution very easily. However, this is only speculation and should be treated cautiously.

The second example is taken from Str 362, published by Neugebauer in MKT I [14: 239-243]. The text contains six problems on a mixture of topics. The first problem is the silver inheritance problem given below and is the only one for which a solution is given; next comes a very broken problem on price-equivalency; then there are two problems dealing with irregular reciprocals, an arithmetic sequence problem involving a broken reed (also given below) and another arithmetic sequence problem involving construction.

The division of silver problem is a bit more sophisticated than the first example, and as in this case the solution procedure is also given, one can see how these types of problems were approached. The problem is simply given, although there are some grave difficulties in understanding exactly how the procedure works. The core of the difficulty is the line that essentially says '1 and 1 is 2'; a lot of trouble is glossed over by that 'essentially'. The conditions of the problem are that 10 brothers share 1 ⅔ mana of silver (in arithmetic progression). The share of the eighth brother is 6 gín. The problem is to determine the difference between the shares of the brothers. It is worth thinking about how a modern mathematician might approach this problem before seeing the rather elegant, in fact, at first sight, downright mysterious, solution given in the text. Note that 1 ⅔ mana is 1,40 mana, or 100 gín. Also, lines in the translation below correspond to grammatical or logical units, not lines of text on the tablet.

> 10 brothers. 1 2/3 mana of silver.
> Brother exceeded brother, but how much he exceeded I do not know.
> The share of the 8th was 6 gín.
> By how much did brother exceed brother?
> You in your proceeding,
> find the reciprocal of 10, the people: it gives 0;6.
> Raise 0;6 to 1 ⅔ mana of the silver: it gives 10.
> Double 10: it gives 20.
> Double 6, the share of the eighth: it gives 12.
> Subtract 12 from 20: it gives 8.
> Let your head hold 8.
> … and 1 below add: it gives 2.
> Double 2: it gives 4
> You add 1 to 4: it gives 5
> Subtract 5 from 10, the people: it gives 5.
> Find the reciprocal of 5: it gives 0;12.
> Multiply 0;12 by 8: it gives 1;36.
> 1;36 is how much brother exceeds brother.

Old Babylonian word problems typically conveyed the solution procedure via a paradigmatic example. As here, each step of the procedure uses values arising from the statement of the problem and previous steps of the solution. A convention is that wherever possible the results of one step are used in the immediately subsequent step.

Where this is not possible, but the result will be needed later, the student is often told to remember the intermediate figure, as here, 'let your head hold 8.' Hence, this instruction indicates that the overall procedure breaks down into two subsections.

Recall, the requirement is to find the difference between the shares of successive brothers. The eldest brother gets the largest share, and each brother down the line gets a smaller share with a constant difference. The first subsection of the problem finds the difference between twice the average share and twice the given brother's share. The average is found by dividing the total by the number of brothers, or, rather, multiplying by the reciprocal of them. This number (10) is doubled; the share of the eighth brother is doubled and the difference found.

For those who are more used to working in an algebraic language, let us give a translation. Label the brothers' shares by $b_0, b_1, \ldots, b_{n-1}$, with b_0 being the smallest, and let d denote the (sought) constant difference. The total $\sum_{i=0}^{n-1} b_i = nb_0 + \frac{n(n-1)}{2}d$ is given. The first steps compute the average, $a = b_0 + \frac{(n-1)}{2}d$, and then twice the average, $2a = 2b_0 + (n-1)d$. Now, if b_k, the share of brother k, is given, we have $b_k = b_0 + kd$, and so

$$\begin{aligned} 2a - 2b_k &= 2b_0 + (n-1)d - 2b_0 - 2kd \\ &= [n-(2k+1)]d \end{aligned}.$$

That is, the number 8 determined at the end of the subsection is a certain multiple of the constant difference. The goal of the second subsection is to determine that multiple.

The next line of the text is where the difficulty of interpreting the problem lies. There is clearly something missing on the tablet, as the line begins 'and 1' and the following word means something like 'lower' or 'below.' In the case of this problem, the given share is that of the eighth brother, so $k = 10 - 8 = 2$, and this is easily found by counting down, or below, the given share to see how many steps there are, 1 and 1.[8] Next, the two steps are doubled, 1 is added and the total subtracted from n, the number of brothers. That is, the factor $n-(2k+1)$ has been computed. The 8 that was previously remembered is now divided by the number obtained and so the problem is solved for d, the amount brother exceeds brother. There is nothing in the procedure that could not be applied equally well with other given data. Quite how these problems were envisioned in the Old Babylonian period is unclear, but there was some quite sophisticated reasoning at work.

Broken Reed problems

In this section we present two 'broken reed' problems. The broken reed was a staple of Old Babylonian mathematics. The reed was used to take some measurement, but pieces

[8] Neugebauer [14, I: 241] and Thureau-Dangin [23: 83], both with a certain uneasiness, proposed a more complicated procedure involving symmetry and shares above and below.

fell off, complicating the result. Generally, the problem was either to find the original length of the reed (an inverse problem) or determine the total distance covered (a direct problem). Not all broken reed problems involve sequences, they were pressed into service for a variety of problems. See [11: 73-75; 84-86] for brief summaries of these problems.

The first problem below is taken from Str 362, the same text as the second division of silver problem above. As noted there, only the statement of the problem is recorded, not the solution procedure. Although the text of the problem is only four lines long, there are some difficulties in reading. Here, we follow Thureau-Dangin's later interpretation [22,23] against Neugebauer's initial reading [14], which led him to propose a geometrical series.

> A reed: 1 kùš.
> 1 šu-si fell off each time until it was all gone.
> How far did I go?
> I went 1 nindan 3 ½ kùš.

The units involved are the kùš, or cubit of 30 šu-si (fingers) and the nindan of 12 kùš. The sense of the problem is that one begins with a (measuring) reed of 1 kùš length and each time a measurement is taken, 1 šu-si breaks off the reed. The procedure is repeated until the reed has reduced to zero, and the total distance measured is desired. In modern notation, what is required is to find the sum (in šu-si) $\sum_{i=0}^{30}(30-i)$, although the solution is given in proper metrological units. Since no details of the solution procedure are given, it is impossible to know how the sum was found – perhaps it was just a tedious addition problem, although the numbers are so carefully chosen that it is hard to resist feeling that a more elegant technique was applied.

The second example of a broken reed sequence problem is taken from AO 6770. This tablet, possibly originally from Larsa, contains five assorted problems: a rectangle problem; a problem involving interest on grain; a problem on finding the original weight of a stone; the volume of bitumen needed to cover a give area, and lastly, a broken reed problem. Unfortunately, it is not only the reed that is broken; the tablet is, too. The statement of the problem can be fairly well restored, but the solution procedure breaks off after a few lines. As this is an inverse problem, the loss of understanding of how it was approached is particularly frustrating. Allowing for some uncertainties in the readings of the last couple of lines, the problem goes as follows:

> I took a reed.
> 1 šu-si fell off each time.
> When it was all gone: 4 kùš.
> What was the original length of the reed?
> The original length of the reed was ½ kùš.
> Find the coefficient of the šu-si. Multiply it by 4 kùš.
> Double it.
> 1, the projection, break in half(?) ……

The problem itself is clear. A reed of unknown length has a finger-length (šu-si) fall off each time a measurement is taken. When the reed is completely used up, the total distance measured is 4 kùš. Find the original length of the reed. In modern terminology, the goal is to solve (converting the given length into šu-si) $\sum_{i=0}^{n}(n-i) = 2{,}0$ for n. The first thing that is unusual about this text is that the solution is stated, and then a procedure is given. The normal practice is to work through the procedure until the solution is given at the end. However, AO 6770 is a very unusual text in many ways: the first problem is one of the rare instances where a general procedure is given instead of a paradigmatic example.

In the case of the problem above, it appears that the first step towards a solution is a scaling to convert the given distance of 4 kùš into units of šu-si. The result is then doubled. Next, the *wasitum*, or projection is taken (and probably broken in half). The term *wasitum* is quite rare in Old Babylonian mathematics, occurring on only four tablets[9], but Høyrup [6] has argued that the term functions as a physical projection, always of magnitude 1, usually from a square. It converts lengths into rectangles so they may be 'added' to areas. As Høyrup puts it, it is 'the width of 1 which transforms a length into an area of equal magnitude.' [6: 298] The use of the *wasitum* is critical in Høyrup's 'cut-and-paste' view of Old Babylonian mathematics. The presence of the term here implies that a quadratic problem is being set up to be solved by the standard completing-the-square procedure (see, for example the discussion in [8] of BM 13901, and the comments below). That in turn implies that the problem is being viewed as an area or quadratic problem, rather than a linear one. When put together with the doubling of the previous step, it is hard to avoid the conclusion that this text provides strong evidence for an awareness of the closed-form formula for the sum of consecutive integers, that is, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Solving a problem by false position

The last example is taken from BM 13901 and shows one way in which false position could be used to solve a simple series problem. BM 13901 was first published by Thureau-Dangin in 1936 [21] and re-edited by Neugebauer in MKT [14]. The text contains a collection of twenty-four problems to do with sides and areas of squares or square fields. The text has been much studied and discussed as it is the classic example for illustrating Høyrup's geometric view of Old Babylonian mathematics. Høyrup has discussed the text many times, included several of the problems in his recent book [8] summarizing his approach and gave a complete new translation in [7]. The first problem is the famous one beginning, "The area and the side of my square I added…" [9] and

[9] The others are BM 13901, VAT 8391 and VAT 8528.

showing the simple cut-and-paste procedure for solving the resulting problem. Although all the problems on the tablet are to do with squares, the problem below, Number 15, does not need the cut-and-paste techniques, showing the more arithmetical origin of these kinds of problems.

> I have added the areas of my four squares: 0;27,5.
> The side, two-thirds, a half, a third of the side.
> You inscribe 1 and 0;40 and 0;30 and 0;20.
> 1 and 1 you multiply: 1.
> 0;40 and 0;40 you multiply: 0;26,40.
> 0;30 and 0;30 you multiply: 0;15.
> 0;20 and 0;20 you multiply: 0;6,40.
> 0;6,40 and 0;15 and 0;26,40 and 1 you add: 1;48,20.
> The reciprocal of 1;48,20 is not detached.
> What do I multiply 1;48,20 by to get 0;27,5: 0;15.
> 0;30 is the square root of 0;15.
> Multiply 0;30 by 1: 0;30 is the first square-side.
> Multiply 0;30 by 0;40: 0;20 is the second square-side.
> Multiply 0;30 by 0;30: 0;15 is the third square-side.
> Multiply 0;30 by 0;20: 0;10 is the fourth square-side.

The total area of the four squares is known and the ratios of the sides of the squares is known, that is, the sides form a descending sequence of the side of the largest square, then two-thirds, half and one third of the large side. The procedure is as follows. Set the side of the largest square to be 1. Then the sides of the other squares are 0;40, 0;30 and 0;15 respectively. Next, determine the sum of the areas of squares of these sides: 1;48,20. To get the ratio of this 'false' area to the actual one, you need to divide by the false area, but the reciprocal of 1;48,20 does not exist (the number is irregular; it is divisible by 13). Instead, the text uses the standard strategy for irregular numbers, determining directly the factor, in this case 0;15. If the ratio of the areas is 0;15, the ratio of the sides is the square root of that, or 0;30. The final step is to multiply each of the 'false' sides by the factor 0;30 to get the correct sides.

Bibliography

1. Civil, M. (1985). 'Sur les "livres d'écolier" à l'époque Paléo-Babylonienne'. In *Miscellanea Babylonica. Mélanges offerts à Maurice Birot* (eds. Durand, J.M. and Kupper, J.-R.) Paris.
2. Englund, R.K. and Nissen, H.J. (1993). *Die lexikalischen Listen der archaischen Texte aus Uruk* (*ATU* 3) Berlin.
3. Friberg, J. (1987-90). 'Mathematik', in *Reallexikon der Assyriologie* VII (ed. D. O. Edzard et al.), Berlin and New York: De Gruyter, 531-585.
4. Friberg, J. (2000). 'Mathematics at Ur in the Old Babylonian period'. *Revue d'Assyriologie* 94, 97-188.

5. Hilprecht, H.V. (1906). *Mathematical, metrological and chronological tablets from the library at Nippur* (BE 20,1). Department of Archaeology, The University of Pennsylvania, Philadelphia.
6. Høyrup, J. (1990). 'Algebra and naive geometry: an investigation of some basic aspects of Old Babylonian mathematical thought'. *Altorientalische Forschungen* 17, 27-69; 262-354.
7. Høyrup, J. (2001). 'The Old Babylonian Square texts - BM 13901 and YBC 4714: Retranslation and analysis.' In *Changing views on ancient Near Eastern Mathematics* (eds. J. Høyrup and P. Damerow) Dietrich Reimer Verlag, Berlin, 155-218.
8. Høyrup, J. (2002). *Lengths, widths, surfaces: A portrait of Old Babylonian algebra and its kin*. Springer, New York.
9. Melville, D.J. (forthcoming). 'The area and the side I added: Some Old Babylonian geometry,' *Revue d'Histoire des Mathématiques*.
10. Muroi, Kazuo (1988). 'Inheritance problems of Babylonian mathematics'. *Historia Scientiarum* 34, 11-19.
11. Nemet-Nejat, K. R. (1993). *Cuneiform mathematical texts as a reflection of everyday life in Mesopotamia* (American Oriental Series 75), New Haven, American Oriental Society.
12. Nemet-Nejat, K.R. (2001). 'Lucky Thirteen'. *Bibliotheca Orientalis* 58, 60-72.
13. Nemet-Nejat, K.R. (2002). 'Square tablets in the Yale Babylonian Collection'. In *Under One Sky: Astronomy and Mathematics in the Ancient Near East* (London, 2001) (eds. J.M. Steele and A. Imhausen) (AOAT 297), Ugarit-Verlag, Münster, 253-281.
14. Neugebauer, O. (1935-37). *Mathematische Keilschrifttexte* I-III (MKT). Berlin, Springer.
15. Neugebauer, O. and Sachs, A. (1945). *Mathematical cuneiform texts* (MCT) (American Oriental Series 29), New Haven: American Oriental Society.
16. Robson, E. (2000). 'Mathematical cuneiform tablets in Philadelphia. I. Problems and calculations'. *SCIAMVS* 1, 11-48.
17. Robson, E. (2001). 'The Tablet House: A scribal school in Old Babylonian Nippur'. *Revue d'Assyriologie* 95, 39-66.
18. Robson, E. (2002). 'More than metrology: Mathematics education in an Old Babylonian scribal school'. In *Under One Sky: Astronomy and Mathematics in the Ancient Near East* (London, 2001) (eds. J.M. Steele and A. Imhausen) (AOAT 297), Ugarit-Verlag, Münster, 325-365.
19. Robson, E. (2003). 'Tables and tabular formatting in Sumer, Babylonia, and Assyria, 2500 BCE-50 CE'. In *The History of Mathematical Tables. From Sumer to Spreadsheets* (eds. M. Campbell-Kelly, M.Croarken, R. Flood and E. Robson) Oxford University Press, Oxford, 18-47.
20. Sachs, A.J. (1947). 'Babylonian mathematical texts I: reciprocals of regular sexagesimal numbers'. *Journal of Cuneiform Studies* 1, 219-240.
21. Thureau-Dangin, F. (1936). 'L'Équation du deuxième degré dans la mathématique babylonienne d'après une tablette inédite du British Museum'. *Revue d'Assyriologie* 33, 27-48.

22. Thureau-Dangin, F. (1936). 'Review of *Mathematische Keilschrifttexte* I-II, by O. Neugebauer'. *Revue d'Assyriologie* 33, 55-61.
23. Thureau-Dangin, F. (1938). *Textes mathématiques babyloniens* (TMB) (Ex Oriente Lux 1), Leiden.
24. Veldhuis, N. (1997). *Elementary Education at Nippur: The Lists of Trees and Wooden Objects*. Docotoral Thesis, University of Groningen (unpublished).
25. Veldhuis, N. (1997-8). 'Review of Cavigneaux, A. Uruk. Altbabylobische Texte aus dem Planquadrat Pe XVI-4/5'. *Archiv für Orientforschung* 44-45, 360-363.