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BABYLONIAN MATHEMATICS
with Special Reference to Recent Discoveries*

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IN a vice-presidential address before Section A of the American Association for the Advancement of Science just six years ago, I made a somewhat detailed survey¹ of our knowledge of Egyptian and Babylonian Mathematics before the Greeks. This survey set forth considerable material not then found in any general history of mathematics. During the six years since that time announcements of new discoveries in connection with Egyptian mathematics have been comparatively insignificant, and all known documents have probably been more or less definitively studied and interpreted. But the case of Babylonian mathematics is entirely different; most extraordinary discoveries have been made concerning their knowledge and use of algebra four thousand years ago. So far as anything in print is concerned, nothing of the kind was suspected even as late as 1928. Most of these recent discoveries have been due to the brilliant and able young Austrian scholar Otto Neugebauer who now at the age of 36 has a truly remarkable record of achievement during the past decade. It

was only in 1926 that he received his doctor's degree in mathematics at Göttingen, for an interesting piece of research in Egyptian mathematics; but very soon he had taken up the study of Babylonian cuneiform writing. He acquired a mastery of book and periodical literature of the past fifty years, dealing with Sumerian, Akkadian, Babylonian, and Assyrian grammar, literature, metrology, and inscriptions; he discovered mathematical terminology, and translations the accuracy of which he thoroughly proved. He scoured museums of Europe and America for all possible mathematical texts, and translated and interpreted them. By 1929 he had founded periodicals called *Quellen und Studien zur Geschichte der Mathematik*² and from the first, the latter contained remarkable new information concerning Babylonian mathematics. A trip to Russia resulted in securing for the *Quellen* section, Struve's edition of the first complete publication of the Golenishchev mathematical papyrus of about 1850 B.C. The third and latest volume of the *Quellen*, appearing only about three months ago,

¹ "Mathematics before the Greeks," *Science*, n.s. v. 71, 31 Jan. 1930, p. 109-121.

² I shall later refer to the two periodicals simply by the words *Quellen*, and *Studien*.

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is a monumental work by Neugebauer himself, the first part containing over five hundred pages of text, and the second part in large quarto format, with over 60 pages of text and about 70 plates. This work was designed to discuss most known texts in mathematics and mathematical astronomy in cuneiform writing. And thus we find that by far the largest number of such tablets is in the Museum of Antiquities at Istanbul, that the State Museum in Berlin made the next larger contribution, Yale University next, then the British Museum, and the University of Jena, followed by the University of Pennsylvania, where Hilprecht, some thirty years ago, published a work containing some mathematical tables. In the Museum of the Louvre are 16 tablets; and then there are less than 8 in each of the following: the Strasbourg University and Library, the Musée Royaux du Cinquantenaire in Brussels, the J. Pierpont Morgan Library Collection (temporarily deposited at Yale) the Royal Ontario Museum of Archaeology at Toronto, the Ashmolean Museum at Oxford, and the Böhl collection at Leyden. Most of the tablets thus referred to date from the period 2000 to 1200 B.C. It is a satisfaction to us to know that the composition of this wonderful reference work was in part made possible by The Rockefeller Foundation. Some two years ago it cooperated in enabling Neugebauer to transfer his work to the Mathematical Institute of the University of Copenhagen, after Nazi intolerance had rendered it impossible to preserve his self respect while pursuing the intellectual life. This new position offered the opportunity for lecturing on the History of Ancient Mathematical Science. The first volume of these lectures,³ on "Mathematics before the Greeks," was published last year, and in it are many references to results, the exact

³ *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, v. 1, *Vorgriechische Mathematik* (Die Grundlehren der mathematischen Wissenschaften, v. 43), Berlin, 1934. Reference to this work will later be made simply by the word *Vorlesungen*.

setting of which are only found in his great source work referred to a moment ago. In these two works, then, we find not only a summing up of Neugebauer's wholly original work, but also a critical summary of the work of other scholars such as Frank, Gadd, Genouillac, Hilprecht, Lenormant, Rawlinson, Thureau-Dangin, Weidner, Zimmern, and many others.⁴ Hence my selection of material to be presented to you to-night will be mainly from these two works. Before turning to this it may not be wholly inappropriate to interpolate one remark regarding Neugebauer's service to mathematics in general. Since 1931 his notable organizing ability has been partially occupied in editing and directing two other periodicals, (1) *Zentralblatt für Mathematik* (of which 11 volumes have already appeared), and (2) *Zentralblatt für Mechanik*, (3 volumes)—a job which of itself would keep many a person fully employed. *Mais, revenons à nos moutons!*

From about 3500 to 2500 years before Christ, in the country north of the Persian Gulf between the Tigris and Euphrates Rivers, the non-semitic Sumerians, south of the semitic Akkadians, were generally predominant in Babylonia. By 2000 B.C. they were absorbed in a larger political group. One of the greatest of the Sumerian inventions was the adoption of cuneiform script; notable engineering works of the Babylonians, by means of which marshes were drained and the overflow of the rivers regulated by canals, went back to Sumerian times, like also a considerable part of their religion and law, and their system of mathematics, except, possibly, for certain details. As to mathematical transactions we find that long before coins were in use the custom of paying interest for the loan of produce, or of a certain weight of a precious metal,

⁴ For the literature of Babylonian mathematics prior to 1929, see my *Bibliography* in the Chace-Manning-Bull edition of the *Rhind Mathematical Papyrus*, v. 2; for later items see K. Vogel's bibliography in *Bayer. Blätter f. d. Gymnasialschulwesen*, v. 71, 1935, p. 16-29.

was common. Sumerian tablets indicate that the rate of interest varied from 20 per cent to 30 per cent, the higher rate being charged for produce. At a later period the rate was $5\frac{1}{2}$ per cent to 25 per cent for metal and 20 per cent to $33\frac{1}{3}$ per cent for produce.⁵ An extraordinary number of tablets show that the Sumerian merchant of 2500 B.C. was familiar with such things as weights and measures, bills, receipts, notes and accounts.

Sumerian mathematics was essentially sexagesimal and while a special symbol for 10 was constantly used it occupied a subordinate position; there were no special symbols for 100 or for 1000. One hundred was thought of as $60+40$ and 1000 as $16 \cdot 60+40$. But in these cases the Sumerian would write simply 1,40 and 16,40;



$= 12 \times 60^2 + 25 \times 60 + 33 = 44733$. Hence the Sumerians had a relative positional notation for the numbers. The word cuneiform means wedgeshaped and the numbers from one to nine were denoted by the corresponding number of wedges, where the Egyptian simply employed strokes. For 10, as we have seen, an angle-shaped sign was used. Practically all other integers were made up of combinations of these in various ways. There is great ambiguity because, for example, a single upright wedge may stand for 1 or 60 or any positive or negative integral multiple of 60. Hence there was a special sign D for 60; D or D or K for 600, the last of which suggests 60×10 ; O for 3600; and C for 36000, again suggesting a product. No special sign for zero in Sumerian times, other than an empty space, has yet been discovered. But by the time of the Greeks



⁵ M. Jastrow, Jr., *The Civilization of Babylonia and Assyria*, Philadelphia, 1915, p. 326, 338; C. H. W. Johns, *Babylonian and Assyrian Laws, Contracts and Letters*, New York, 1904, p. 251, 255-256. See also D. E. Smith, *History of Mathematics*, vol. 2, 1925, p. 560.

$+0+33=43233$, the sign Δ being for zero. But to matters of numeral notation we shall make no further reference, except to remark that the Babylonians thought of any positive integer $a = \sum c_n \cdot 60^n$, and in the form

$$a = \dots c_2 c_1 c_0 c_{-1} c_{-2} \dots$$

This may not, of course, correspond to what we call integers. By means of negative values of n , fractions were introduced.

Babylonian multiplication tables are very numerous and are often the products of a certain number, successively, by 1, 2, 3 \dots 20, then 30, 40 and 50. For example, on tablets of about 1500 B.C. at Brussels are tables of 7, 10, $12\frac{1}{2}$, 16, 24, each multiplied into such a series of numbers. There are various tablets giving the squares of numbers from 1 to 50, and also the cubes, square roots and cube roots of numbers. But we must be careful not to assume too much from this statement; the tables of square roots and cube roots were really exactly the same as tables of squares and cubes, but differently expressed. In the period we are considering the Egyptian really had nothing to correspond to any of these tables, nor do we know that even the conception of cube root was within his ken. Until two years ago it was a complete mystery why the Babylonians had tables of cubes and cube roots, but finally a tablet in the Berlin Museum gave a clue. This is a table of $n^3 + n^2$, for $n = 1$ to 30. Certain problems on British Museum tablets were found to lead to cubic equations of the form $(ux)^3 + (ux)^2 = 252$. Hence Neugebauer reasoned in his article of 1933 in the *Göttingen Nachrichten* that the purpose of the tablet in question was to solve cubic equations in this "normal form." He contended that it was within the power of the Babylonians, by a linear transformation $z = x + c$, to reduce a four-term cubic equation $x^3 + a_1x^2 + a_2x + a_3 = 0$ to $z^3 + b_1z^2 + b_2 = 0$. Multiplying this equation by $1/b_1^3$ we have at once (on setting $z = b_1w$, and $a = -b_2/b_1^3$) the normal form

$$w^3 + w^2 = a.$$

Neugebauer's theory as to the possibility of such a reduction is in part based on problems to which I shall later refer. Up to the present, however, Neugebauer has found no four-term cubic equation solved in this way. And indeed in these same British Museum tablets are two problems which lead naturally to such equations but are solved by a different method.⁶

Neugebauer feels that *tables are the foundation of all discussion of Babylonian mathematics*, that more tables, such as the one to which we have just referred, are likely to be discovered, and to illuminate other mathematical operations. There are many tables of parallel columns of integers such as

| | |
|---|-------|
| 2 | 30 |
| 3 | 20 |
| 4 | 15 |
| 5 | 12 |
| 6 | 10 |
| 8 | 7, 30 |
| 9 | 6, 40 |

which is nothing but a table of reciprocals $1/n = \bar{n}$ in the sexagesimal system. $n \cdot \bar{n}$ is always equal to 60 raised to 0, or some positive or negative integral power. It is notable that in the succession of numbers chosen, the divisor $n = 7$ does not appear, the reason being that there is no integer \bar{n} such that the product is the power of 60 indicated. Hence every divisor, a , with a corresponding \bar{n} must be of the form $a = 2^x \cdot 3^y \cdot 5^z$. All such reciprocals are called regular; and such reciprocals as of 7 and 11 irregular.⁷ When irregular numbers appear in tables the statement is made that they do not divide.

Some of these tables are extraordinary in their complexity and extent. One tablet in the Louvre, dating from about the

⁶ Compare Göttingen, *Nachrichten, Math.-phys. Kl.*, 1933, p. 319; also *K. Danske Videnskabernes Selskab, Mathem.-fysiske Meddelelser*, v. 12, no. 13, p. 9. Also *Quellen*, v. 3, part 1, p. 200-201, 210-211.

⁷ It is easy to approximate to $1/7$, e.g. $7/28 = ; 8, 45$; $13/90 = ; 8, 40$, etc., but there is no case known where this was done.

time of Archimedes, has nearly 250 reciprocals of numbers many of them six-place, and some seven. For example, here is the second last entry for a six-place number:⁸ 2, 59, 21, 40, 48, 54 20, 4, 16, 22, 28, 44, 14, 57, 40, 4, 56, 17, 46, 40 that is, the product of $(2 \times 60^5 + 59 \times 60^4 + \dots + 54) \times (20 \times 60^{13} + 4 \times 60^{12} + \dots + 40) = 60^{19}$.

The object of a table of reciprocals is to reduce division to multiplication since b/a equals b multiplied by the reciprocal of a .

I referred a few moments ago to one-figure tables of squares (that is, the squares of numbers from 1 to 60). In a tablet of the Ashmolean Museum at Oxford is the only example at present known of a two-figure table of squares.⁹ This dates from about 500 B.C. The tablet is of further interest from the fact that on it are several examples of the sign for zero, e.g.

$$(15, 30)^2 = 4, \cdot, 15$$

$$(39, 30)^2 = 26, \cdot, 15.$$

The latter is equivalent to $2370^2 = 5,616,900$

Among table-texts are also certain ones involving exponentials. From Neugebauer's volume of Lectures we may easily gain the impression¹⁰ that these are tables for c^n , $n = 1$ to 10, for $c = 9$, $c = 16$, $c = 100$, and $c = 225$. On turning, however, to his work published three months ago we find that the tables in question are on Istanbul tablets, which are in very bad condition, so that for $c = 16$ there is not a single complete result; for $c = 9$ there are only three complete results, and similarly for the others. Enough is present however to show that the original was probably at one time as described.

One use of such tablets is in solving problems of compound interest. For example in a Louvre text dating back to

⁸ *Quellen*, v. 3, part 1, p. 22.

⁹ *Quellen*, v. 3, part 1, p. 72-73 and part 2, plate 34.

¹⁰ *Vorlesungen*, p. 201; *Quellen*, v. 3, part 1, p. 77-79 and part 2, plate 42.

2000 B. C. is a question as to how long it would take for a certain sum of money to double itself at 20% interest.¹¹ The problem here, then, is to find x in the equation

$$(1; 12)^x = 2.$$

The answer given is 4-0; 2, 33, 20=3; 57, 26, 40 years, not so very different from the more accurate result 3; 48. That is, from $(1; 12)^4 = 2$; 4, 24, 57, 36 4 was found too large, giving a quantity greater than 2. How the amount to subtract was discovered is not indicated in the text, and can not now be surmised. This is a conspicuous example of a solution by the Babylonians of an equation of the type $a^x = b$ where x was not integral.

Both in the Berlin Museum and in Yale University are tablets with other problems in compound interest. If for no other reason than to point out that five-year plans are not wholly a modern invention I may refer to a problem in a Berlin papyrus, the transcription and discussion of which occupies 16 pages of Neugebauer's new book.¹² As yet I have not mastered all the discussion of this problem, but certain facts can be stated with assurance. There is a very curious combination of simple and compound interest which is naturally suggestive of what may have been customary in old Babylonia. If P is an amount of principal, r is the rate of interest per year (here 20%), and we suppose that through a five-year period P accumulates at simple interest it will amount to $2P$ at the end of the first five-year period. This amount $2P$ is then put at interest in the same way for a second five-year period and the principal is again doubled to 2^2P . The amount of capital at the end of any year is therefore given by the formula

$$A = 2^n P(1 + rm)$$

where $0 \leq m < 5$, and n is the number of five-year periods. One of the problems is: How many five-year periods will it take

¹¹ *Quellen*, v. 3, part 2, p. 37-38, 40-41.

¹² *Quellen*, v. 3, part 1, p. 351-367, and part 2, plates 29, 32, 54, 56, 57; *Vorlesungen*, p. 197-199.

for a given principal P to become a given sum A ? The particular case when $m = 0$ gives us the equation $A = 2^n P$. In modern notation $n = \log_2 A/P$. Now Neugebauer suggests as a possible theory in explanation of the text that something equivalent to logarithms to the base 2 was here used. In the problem $P = 1$, $A = 1,4$ whence $n = 6$.

Two other suggestive problems of the Babylonians, involving powers of numbers are in a Louvre tablet of about the time of Archimedes.¹³ We have here 10 terms of a geometric series in which the first term is 1, and 2 the constant multiplier; the sum is given correctly,

$$\sum_{i=0}^9 2^i = 1 + 2 + 2^2 + \dots + 2^9 = 1023.$$

But what is of special interest is the apparent suggestion as to how this number 1023 was obtained. On the tablet it is stated that it is the sum of $511 = 2^9 - 1$, and $512 = 2^9$. That is

$$1023 = 2^9 + 2^9 - 1 = 2 \cdot 2^9 - 1 = 2^{10} - 1.$$

Does this imply a knowledge of Euclid's formula leading to the sum of the ten terms of the geometric progression as, $(2^{10} - 1)/(2 - 1)$?

On the same tablet is the following

$$1 \cdot 1 + 2 \cdot 2 + \dots + 10 \cdot 10 \\ = (1 \cdot 1/3 + 10 \cdot 2/3) \cdot 55 = 385.$$

That is, we have the sum of the squares of the first 10 integers, and this sum is the product of two integers, one of which is 55, the sum of the first 10 integers. In general terms this relation may be stated

$$\sum_{i=1}^n i^2 = (1 \cdot 1/3 + n \cdot 2/3) \sum_{i=1}^n i.$$

Now if we set $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$, a formula known to the Pythagoreans, we have $\sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1)$. This formula is practically equivalent to one known to Archimedes.

¹³ *Quellen*, v. 3; part 1, p. 96-97, 102-103 and part 2, plate 1; *Studien*, v. 2, 1932, p. 302-303.

Turning back to tables for a moment one finds a word for subtraction, *lal*; 19 is 20 *lal* 1, 37 is 40 *lal* 3; a *lal* $b = a - b$. Neugebauer refers to a late astronomical text in which *before* each of 12 numbers the words *tab* and *lal* (plus and minus)¹⁴ are placed, suggesting the arrangements of points above and below a line which lie on a wave-shaped curve. This seems extraordinary. Neugebauer promised more about the matter in the third volume of his Lectures which is to deal with mathematical astronomy.

It is also a matter of great historical interest that, in at least three different problems in simultaneous equations in two unknowns, *negative numbers occur as members*. These examples are in Yale University texts,¹⁵ and it is very noteworthy that such conceptions were not current in Europe, even 2500 years later.

As a point of departure for certain other things let us now consider some geometrical results known to the Babylonians. There will of course be no misunderstanding when I state general results. These simply indicate operations used in many *numerical* problems of the Babylonians.

1. The area of a *rectangle* is the product of the lengths of two adjacent sides.

2. The area of a *right triangle* is equal to one half the product of the lengths of the sides about the right angle.

3. The sides about corresponding angles of *two similar right triangles* are proportional.

4. The area of a *trapezoid* with one side perpendicular to the parallel sides is one-half the product of the length of this perpendicular and the sum of the lengths of the parallel sides.

5. The perpendicular from the vertex of an *isosceles triangle* on the base, bisects the base. The area of the triangle is the product of the lengths of the altitude and half the base.¹⁶ Indeed the Babylonians

¹⁴ *Vorlesungen*, p. 18.

¹⁵ *Quellen*, v. 3, part 1, p. 387, 440, 447, 455, 456, 463, 470, 474; and part 2, plates 23, 48, 59.

¹⁶ *Quellen*, v. 3, part 2, p. 43, 46-47, 50-51, and part 1, p. 97, 104.

would probably think of the area of a triangle, other than right or isosceles, as the product of the lengths of its base and altitude—an easy deduction from two adjacent, or overlapping, right triangles. A large rectilinear area portrayed in a Tello tablet, in the Museum at Istanbul,¹⁷ was calculated by dividing it up into 15 parts: 7 right triangles, 4 rectangles (approximately), and 4 trapezoids.

6. The *angle in a semicircle* is a right angle, a result till recently first attributed to Thales of Miletus, who flourished 1500 years later.

7. $\pi = 3$, and the *area of a circle* equals one twelfth of the square of the length of its circumference (which is correct if $\pi = 3$). $A = \pi r^2 = (2\pi r)^2/4\pi$.

8. The *Pythagorean theorem*, a result entirely unknown to the Egyptian.¹⁸

9. The volume of a *rectangular parallelepiped* is the product of the lengths of its three dimensions, and the volume of a *right prism* with a trapezoidal base is equal to the area of the base times the altitude of the prism. Such a volume as the latter would be considered in estimating the amount of earth dug in a section of a canal. In a British Museum tablet the volume of a solid equivalent to that cut

¹⁷ A. Eisenlohr, *Ein altbabylonischer Feldplan*, Leipzig, 1896. J. Oppert, Académie d. Inscriptions et Belles-Lettres, *Comptes Rendus* s. 4, v. 24, 1896, p. 331-348; also in *Revue d'Assyriologie et d'Archéologie Orientale*, v. 4, 1897, p. 28-33. F. Thureau-Dangin, *Revue d'Assyriologie*, v. 4, 1897, p. 13-27.

¹⁸ After many mistatements by mathematical historians it was an Egyptologist, the late T. E. Peet, in his *Rhind Mathematical Papyrus* (London, 1923, p. 31-32) who brought out the fact that there is not one scrap of evidence that the Egyptians knew the Pythagorean theorem, even in the simple 3-4-5 case. He gave also interesting new information about the *harpedonaptai*, or rope stretchers, referred to by Democritus. It is of course true that there are problems involving the relations of such numbers as 8, 6, and 10, as in Berlin Papyrus 6619 (about 1850 B. C.): Distribute 100 square ells between two squares whose sides are in the ratio 1 to $\frac{1}{2}$. The same equations arise in problem 6 of the Golenishev papyrus: Given that the area of a rectangle is 12 arurae and the ratio of the lengths of the sides $1:\frac{1}{2}$, find the sides; see also the Kahun papi, ed. by Griffith (1898).

off from a rectangular parallelepiped by a plane through a pair of opposite edges is given correctly as half that of the parallelepiped.¹⁹

10. The volume of a *right circular cylinder* is the area of its base times its altitude.

11. The volume of the *frustum of a cone* is equal to its altitude multiplied by the area of its median cross-section.²⁰

12. The volume of the *frustum of a cone*, or of a square pyramid, is equal to one-half its altitude multiplied by the sums of the areas of its bases. [Contrast this approximation to the volume of a frustum of a square pyramid with the exact formula known to the Egyptians of 1850 B. C., $V = \frac{1}{3}h(a_1^2 + a_1a_2 + a_2^2)$, where a_1, a_2 are the lengths of sides of the square bases, and h the distance between them.] On the other hand Neugebauer believes that the Babylonians also had an exact value for the volume of the frustum of a square pyramid, namely²¹

$$V = h \left[\left(\frac{a_1 + a_2}{2} \right)^2 + \frac{1}{3} \left(\frac{a_1 - a_2}{2} \right)^2 \right];$$

concerning the second term there has been more than one discussion.

Practically all of these results are in British Museum texts of 2000 B. C.

That the Pythagorean theorem was known to the Babylonians of 2000 B. C. is certain from the following problems of a British Museum text:²² (1) To calculate the length of a chord of a circle from its *sagitta* and the circumference of the circle; and (2) To calculate the length of the *sagitta* from the chord of a circle, and its circumference. If c be the length of the chord, a of its *sagitta* and d of the diameter

¹⁹ *Quellen*, v. 3, part 2, p. 43, 47, 52.

²⁰ *Studien*, v. 1, 1929, p. 86-87; *Vorlesungen*, p. 171; *Quellen*, v. 3, part 1, p. 176.

²¹ *Studien*, v. 2, 1933, p. 348-350; *Vorlesungen*, p. 171; *Quellen*, v. 3, part 1, p. 150, 162, 187-188. Heron of Alexandria (second century A. D.?) found the volume of such a pyramid, for which $a_1 = 10, a_2 = 2, h = 7$ (*Heronis Alexandrini Opera quae supersunt omnia*, Leipzig, v. 5, 1914, p. 30-35), every step being equivalent to substituting in this formula.

²² *Studien*, v. 1, 1929, p. 90-92.

(one-third of the circumference) of the circle, the formulae used are evidently

$$c = \sqrt{d^2 - (d - 2a)^2}$$

$$a = \frac{1}{2} [d - \sqrt{d^2 - c^2}].$$

Now every step of the numerical work is equivalent to substitution in these formulae.

The same is true of the following problem in another British Museum tablet.²³ A beam of given length l was originally upright against a vertical wall but the upper end has slipped down a given distance h , what is the distance d of the other end from the wall? Each step is equivalent to substitution in the formula

$$d = \sqrt{[l^2 - (l - h)^2]}$$

and then follows the converse problem, given l and d to find h ,

$$h = l - \sqrt{l^2 - d^2}.$$

In these problems a, c, d, h , and l are all integers.

A third problem involving the use of the Pythagorean theorem is one on a Louvre tablet of the Alexandrine period:²⁴ Given in a rectangle that the sum of two adjacent sides and the diagonal is 40 and that the product of the sides is 120. The sides are found to be 15 and 8 and the diagonal 17.

There are, however, various problems in Babylonian mathematics where square roots of non-square numbers, such as 1700, are discussed. In this particular case the problem, on an Akkadian tablet of about 2000 B.C., is to find the length of the diagonal of a rectangle whose sides are ;40 and ;10. It is worked out twice, as if by two approximation formulae.²⁵ If the lengths of the diagonal and sides of a rectangle are respectively d, a , and b , $d = \sqrt{a^2 + b^2}$ and the approximation formulae are:

²³ *Quellen*, v. 3, part 2, p. 53.

²⁴ *Studien*, v. 2, 1932, p. 294; *Quellen*, v. 3, part 1, p. 104.

²⁵ *Studien*, v. 2, 1932, p. 291-294; *Vorlesungen*, p. 33-36; *Quellen*, v. 3, part 1, p. 279-280, 282, 286-287, and part 2, plates 17, 44.

$$(1) \quad d = a + \frac{b^2}{2a};$$

$$(2) \quad d = a + 2ab^2.$$

The first of these is equivalent to the one employed several times, two thousand years later, by Heron of Alexandria, in his *Metrica*.

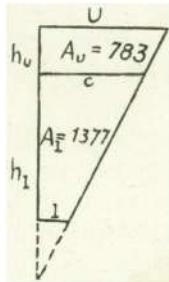
On the other hand the dimensions of the second formula are incorrect. After considerable calculation Neugebauer shows that a correct and good approximation to d is given by the following

$$(3) \quad d = a + \frac{2ab^2}{2a^2 + b^2}.$$

Since, in the particular problem in question, $1/(2a^2 + b^2) = 12/11$ that is, almost unity, Neugebauer has surmised that the equivalent of this third formula may have been used.

The Heron approximation formula was also used by the Babylonians to find²⁶ $1\frac{5}{8}$ for $\sqrt{2}$ and $17/24$ for $1/\sqrt{2}$.

In a British Museum text of about 2000 B. C. there is an interesting attempt to approximate $\sqrt{2\frac{1}{2}}$ by a step equivalent to that of seeking the solutions of the Diophantine equation²⁷ $y^2 + 22\frac{1}{2} = x^2$. When $x = 5$, $y = \sqrt{2\frac{1}{2}}$. The values $y = 1\frac{1}{2}$, $x = 5\frac{1}{4}$ are found in the text. One readily finds also $y = 1\frac{1}{8}$, $x = 4\frac{7}{8}$ so that the required y is between $1\frac{1}{8}$ and $1\frac{1}{2}$. Babylonian tables of squares might well give much closer approximations.



²⁶ *Studien*, v. 2, 1932, p. 294-295; *Vorlesungen*, p. 37; *Quellen*, v. 3, part 1, p. 100, 104, and part 2, plate 1.

²⁷ *Studien*, v. 2, 1932, p. 295-297, 309; *Quellen*, v. 3, part 1, p. 172.

Two of the geometrical theorems referred to, a few moments ago, are employed in the solution of the following problem of a Strasbourg tablet.²⁸ Consider two adjacent trapezoids, sections of the same right triangle and with a common side of length c as in the figure. The upper area of height h_u (between u and c) is given as 783; the lower area of height h_l (between c and l) is 1377. It is further given that

$$(1) \quad h_l = 3h_u$$

$$(2) \quad u - c = 36$$

Then by applying the theorems mentioned

$$(3) \quad h_u \cdot \frac{u+c}{2} = 783$$

$$(4) \quad h_l \cdot \frac{c+l}{2} = 1377$$

$$(5) \quad u - c = (1/3)(c - l)$$

five equations from which the five unknown quantities are found.

There are many similar problems, one, of a group, leading to ten equations in ten unknowns. This is in connection with the division of a right triangle (by lines parallel to a side) into six areas of equal altitudes, while their areas are in arithmetic progression.²⁹ This problem seems to show mathematics studied for its own sake, just as problem 40 of the Rhind papyrus suggested a similar thought there.

Consider now another Strasbourg problem, of a different type, leading to a quadratic equation:³⁰ The sum of the areas of two squares is a given area. The length (y) of the side of one square exceeds a given ratio (α/β) of the length (x) of the side of the other square, by a quantity d . The problem is to find x and y . Here

$$x^2 + y^2 = A,$$

$$y = \frac{\alpha}{\beta} x - d.$$

²⁸ *Studien*, v. 1, 1929, p. 67-74; *Quellen*, v. 3, part 1, p. 259-263.

²⁹ *Quellen*, v. 3, part 1, p. 253; *Studien*, v. 1, 1929, p. 75-78; *Vorlesungen*, p. 180-181.

³⁰ *Studien*, v. 1, 1930, p. 124-126; *Quellen*, v. 3, part 1, p. 246-248.

If we set $x = X\beta$ it may be readily shown that we are led to the equation

$$X^2 - \frac{2d\alpha}{\alpha^2 + \beta^2} X - \frac{A - d^2}{\alpha^2 + \beta^2} = 0$$

whence

$$X = \frac{1}{\alpha^2 + \beta^2} \left\{ d\alpha + \sqrt{d^2\alpha^2 + (\alpha^2 + \beta^2)(A - d^2)} \right\}$$

Now every step of the solution of this problem is equivalent to substitution in this formula.

There are scores of problems which prove this amazing fact, that the Babylonians of 2000 B. C. were familiar with our formula for the solution of a quadratic equation. Until 1929 no one suspected that such a result was known before the time of Heron of Alexandria, two thousand years later.

In general only the positive sign before the radical in the solution of a quadratic equation is to be considered; but in the following problem³¹ (because of its nature) both roots are called for. The problem on a Berlin tablet deals with the dimensions of a brick structure of given height h , of length l of width w , and of given volume v . The exact nature of the structure is not clear but it is given that $v/a = hlm$, where $1/a$ is a given numerical factor. $l + m$ is also a given quantity S ; it is required to find l and m . They are evidently roots of the quadratic equation

$$X^2 - SX + \frac{v}{ah} = 0;$$

when l and m are given by

$$\frac{S}{2} \pm \sqrt{\left(\left(\frac{S}{2}\right)^2 - \frac{v}{ah}\right)}.$$

The upper sign gives the required value for l and the lower for m . Of course both roots are positive.

On another Berlin tablet³² is a problem divorced from geometrical connections but

³¹ *Quellen*, v. 3, part 1, p. 280-281, 283-285.

³² *Quellen*, v. 3, part 1, p. 350-351; *Vorlesungen*, p. 186-187.

which may possibly illustrate another point of interest. Two unknowns y_1, y_2 are connected by relations

$$(1) \quad y_1 - \frac{\alpha}{\beta} (y_1 + y_2) = D$$

$$(2) \quad y_1 y_2 = 1$$

where α, β, D are given, $\beta > \alpha$. New variables are then introduced,

$$(3) \quad x_1 = (\beta - \alpha)y_1, \quad x_2 = \alpha y_2$$

whence $x_1 - x_2 = \beta D, x_1 x_2 = \alpha(\beta - \alpha)$. From the resulting quadratic equation

$$X^2 - \beta D X - \alpha(\beta - \alpha) = 0$$

x_1 and $-x_2$ are found to be,

$$\pm \frac{\beta D}{2} + \sqrt{\left[\left(\frac{\beta D}{2}\right)^2 + \alpha(\beta - \alpha)\right]},$$

y_1 and y_2 are then found from (3). Neugebauer emphasizes that here, and in other texts we have a transformation of a quadratic equation to a *normal form* with unity as coefficient of the squared term. And also we have another example of an equation in which both roots are positive and the double sign before the radical is taken in solving the question.

We have now considered Babylonian solutions of simultaneous equations, exponential equations, quadratic equations, and cubic equations. Before giving examples leading to equations of higher degree some general remarks may be made about 17 of the 35 mathematical tablets at Yale.³³ In size they are from 9.5×6.5 cm. to 11.5×8.5 cm. They belong to series and contain the enunciation of problems systematically arranged. No solutions are given. On one tablet there are 200 problems and on the seventeen over 900. Since only a few tablets have been preserved there must have been thousands of problems in the original series.

To give an idea of what is meant by problems being arranged in a series it may

³³ *Quellen*, v. 3, part 1, p. 351-516, and part 2, plates 36, 37, 57-59, p. 60-64; *Studien*, v. 3, 1934, p. 1-10.

be noted that on one tablet are 55 problems of the type to find x and y , given³⁴

$$\begin{cases} xy = 600 \\ (ax + by)^2 + cx^2 + dy^2 = B \end{cases}$$

where some coefficients can be zero. The first equation is the same for all of these problems. The second equations for the first seven problems are as follows:

$$\begin{array}{l} 1. \quad (3x)^2 + y^2 = 8500 \\ 2. \quad \quad + 2y^2 = 8900 \\ 3. \quad \quad - y^2 = 7700 \\ \hline 4. \quad (3x + 2y)^2 + x^2 = 17800 \\ 5. \quad \quad + 2x^2 = 18700 \\ 6. \quad \quad - x^2 = 16000 \\ 7. \quad \quad - 2x^2 = 15100 \end{array}$$

Problems 48 and 49 are

$$\begin{aligned} [3x + 5y - 2(x - y)]^2 - 2y^2 &= 28100 \\ + x^2 + y^2 &= 30200. \end{aligned}$$

The solution of all the equations leads at once to a biquadratic equation which is a quadratic equation in x^2 .

On another tablet, however, are problems of the type³⁵

$$\begin{aligned} xy &= A \\ a(x + y)^2 + b(x - y) + C &= 0 \end{aligned}$$

which leads to the most general form of biquadratic equation (if $d = c + 2aA$)

$$x^4 + \frac{b}{a}x^3 + \frac{d}{a}x^2 - \frac{bA}{a}x + A^2 = 0.$$

The second equation of one of the problems in this group is

$$\frac{1}{5}(x + y)^2 - 60(x - y) = -100,$$

one of the extraordinary examples of a negative number in the right hand member, to which we have already referred.

Problems on another tablet³⁶ lead to the most general cubic equation. How the Babylonians found the solution of such

equations is unknown. It is true that $x = 30$, $y = 20$ gives the solution of every one of these, and of hundreds of problems in other series; Neugebauer believes, however, that it is nonsensical to imagine that such values were merely to be guessed (*Quellen*, v. 3, part 1, p. 456).

There are problems about measurement of corn and grain, workers digging a canal, interest for loan of silver, and more problems like the Strasbourg texts where algebraic questions are derived from consideration of sections of a triangle. Neugebauer concludes the second part of his great work with an italicized statement to the effect that the Strasbourg and Yale texts prove that the chief importance of Babylonian mathematics lies in algebraic relations—not geometric.

In his work of thirty years³⁷ ago Hilprecht was guilty of more than one disservice to truth. One such was his great emphasis on mysticism in Babylonian mathematics, its association with what he called "Plato's number," $60^4 = 12,960,000$. In spite of the protests of contemporary scholars such ideas were widely disseminated. We have noted enough to realize that such an idea is purest bunk—rather freely to translate Neugebauer's expression.³⁸

While it has been possible for me to draw your attention to only a few somewhat isolated facts, I trust that you have received the impression that in Babylonian algebra of 4000 years ago we have something wonderful, *real* algebra without any algebraic notation or any actual setting forth of general theory. And if all this was known in 2000 B. C., how far back must we go for the beginnings of the Sumerian mathematics, simple arithmetic operations? Probably back to 3000 B. C. at least.

³⁷ *Mathematical, Metrological and Chronological Tablets, from the Temple of Nippur* (Univ. Pennsylvania), s.A., Cuneiform Texts, v. 20, part 1, Philadelphia, 1906.

³⁸ *K. Danske Videnskabernes Selskab, Matem.-fysiske Meddelelser*, v. 12, no. 13, 1934, p. 5-6.

³⁴ *Quellen*, v. 3, part 1, p. 418-420.

³⁵ *Quellen*, v. 3, part 1, p. 455-456.

³⁶ *Quellen*, v. 3, part 1, p. 402.

One thing which is of great interest in the study of Egyptian and Babylonian mathematics is, that we handle and study the actual documents which go back to those days, four thousand years ago. Contrast with this the way in which we learn of Greek mathematics. Even in the case of such a widely used work as Euclid's Elements, there is not a single manuscript which is older than 1200 years after Euclid lived, that is about a thousand years ago.

Eight years of work by a young genius standing on the shoulders of great pioneering scholars, have in extraordinary fashion greatly advanced the frontiers of our knowledge of Babylonian mathematics. One can not help feeling that the inspiration of such achievement will cause more than one man to shout, "Let knowledge grow from more to more," as *he* too joins in the endless torch race to "pass on the deathless brand From man to man."

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