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Quadratic Equations in the Susa Mathematical Text No. 21

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I Introduction

In 1933 a team of French archaeologists found some mathematical tablets at Susa, the capital of the ancient kingdom of Elam located in the west of Iran. They must have recognized at a glance that the tablets were very important, for there were a few geometrical figures on them which had not been found in other mathematical tablets; that is, an equilateral triangle inscribed in a circle, a regular hexagon, and a regular heptagon. Though important, the tablets were not published for a long time afterwards. In fact, F. Thureau-Dangin (1872–1944), an eminent French assyriologist who engaged in deciphering mathematical cuneiform texts, did not refer to the tablets at all. In 1961 the texts of the Susa mathematical tablets were finally published by E. M. Bruins and M. Rutten under the title of *Textes mathématiques de Suse*.[1]

The Susa mathematical texts are written in the Akkadian language on twenty-six tablets which probably belong to the late Old Babylonian period (the sixteenth century B.C.). The technical expressions and the mathematical contents of the texts are both very distinctive. An example of the peculiar expressions is the Akkadian verb *alākum* (to go) used in the sense of "multiply" uniquely in mathematical texts of the Old Babylonian period. Another peculiarity is the occurrence of primitive linear indeterminate equations, which I have pointed out elsewhere.[2] These peculiarities of the Susa mathematical texts signify that they are very important for the history of Babylonian mathematics.

Since the Susa mathematical text No. 21 is badly damaged, especially at the beginning where the problem is stated, its mathematical purport is very difficult

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[1] E. M. Bruins et M. Rutten, *Textes mathématiques de Suse* [= TMS], (1961). Bruins informed me that Thureau-Dangin, who had intended to study the Susa mathematical texts, had entrusted him with their publication, asking him not to show the tablets to Neugebauer. Although we cannot believe everything he said, it is true that Thureau-Dangin was on bad terms with Neugebauer in those days.

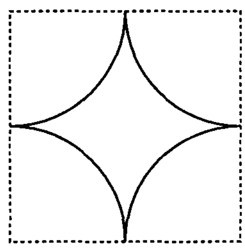
[2] K. Muroi, A New Interpretation of the Susa Mathematical Text No. 7, *Studies in Babylonian Mathematics*, No. 2 (1992), pp. 13–19.

to understand.[3] In fact no satisfactory interpretation has been given to it yet. Fortunately, however, more than half of the calculation process survives and the meanings of certain key geometrical terms are understandable through another Susa mathematical text.

In this paper I shall attempt to restore as many damaged lines as possible and to clarify which equations are dealt with in the text.

II The geometric figure called *apsamikkum*

The Old Babylonian geometric term, *apsamikkum*, denotes a plane figure which consists of four quadrants and resembles an asteroid in form (cf. Fig. 1).



apsamikkum

Figure 1

Since we have no modern term corresponding to the *apsamikkum*, I will render it, following Oppenheim and others, as "regular concave-sided tetragon"[4] in the translation below for the sake of convenience.

Concerning the *apsamikkum*, three constants occur in the Susa mathematical text No. 3 (a table of constants), lines 22–24:[5]

Transliteration

22. 26,40 igi-gub *šà a-pu-sà-am-mi-ki*
23. 1,20 bar-dá *šà a-pu-sà-mi-ki*
24. 33,20 *pi-ir-ku šà a-pu-sà-mi-ki*

[3]TMS, pp. 108–111.

[4]A. L. Oppenheim and others, *The Assyrian Dictionary*, Vol. 1, A, part II (1968), pp. 192–193.

[5]TMS, pp. 25–34. Bruins and Rutten's transliteration, BAR-*ta* (*ṣilipta*) in line 23, is not correct. Cf. K. Muroi, Reexamination of Susa Mathematical Text No. 3: Alleged Value $\pi \fallingdotseq 3\ 1/8$, *Historia Scientiarum*, Vol. 2-1 (1992), pp. 45–49.

Translation

22. 0;26,40 is the (area) constant of a regular concave-sided tetragon.
23. 1;20 is the diagonal of a regular concave-sided tetragon.
24. 0;33,20 is the transverse line of a regular concave-sided tetragon.

These lines denote in order the area of an *apsamikkum*, the length of whose quadrant is 1, the longest and the shortest distances between two points on the curve (cf. Fig. 2). Since in Babylonian mathematics the area of a circle whose circumference is c is calculated by means of the formula, $0;5c^2$, the area of the above *apsamikkum* is:

$$1;20^2 - 0;5 \cdot 4^2 = 0;26,40,$$

where 1;20 ($= 1/3 \cdot 4$) is the diagonal of the *apsamikkum*. Both 0;26,40 and 1;20 occur in our text also. The constant, 33,20, is obtained as follows:

$$\begin{aligned} 1;20 \cdot \sqrt{2} - 1;20 &\approx 1;20 \cdot 1;25 - 1;20 \\ &= 1;20 \cdot 0;25 \\ &= 0;33,20. \end{aligned}$$

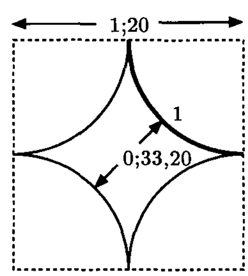


Figure 2

III The Susa mathematical text No. 21

In my judgment this text consists of two similar problems. The first problem is comparatively well preserved, but the second, which must have started on the obverse and continued on the reverse, is mostly lost. Both of them deal with certain quadratic equations.

TMS no. 21 transliteration[6]

Obverse

1. [... ...] ... [5-t]a-àm *uṣ*$_4$*-ṣa-am-m*[*a*... ...]
2. [... ...*a*]*-pu-s*[*à-mi-ku*]*m* sag *i-na li-ib-bi il-*[*la*(?)*-wi*(?)]
3. [35] a-šà dal-ba-na nigin-[*i*]*a mi-nu* {*nu*}
4. [za-e] 5 *mi-i*[*s*]*-si*$_{20}$*-ta a-na* 2 nigin^{sic} 10 *ta-mar*
5. [10] nigin 1,40 *ta-mar* 1,40 *i-na* 35 zi 32^{sic},20 *ta-mar*
6. 1 *a-pu-sà-mi-ka* gar 1,20 dal *šà a-pu-sà-mi-ki* gar
7. 1,20 *a-na* 1 *i-ší-ma* 1,20 *ta-mar* 1,20 *ki-ma* a-šà gar
8. *tu-úr* 1,20 nigin 1,46,40 *ta-mar* 1 nigin 1 *ta-mar*
9. 1 *a-na* 26,40 igi-gub *a-pu-sà-mi-ki i-ší-ma*
10. 26,40 *ta-mar* 26,40 *i-na* 1,46,40 zi
11. 1,20 *ta-mar* 1,20 *a-na* 33,20 a-šà dal-ba-na *i-ší-ma*
12. [4]4,26,40 *ta-mar* 1,[20] *a-na* 10 *mi-is-si*$_{20}$*-ti i-ší-ma*
13. [13,20] *ta-mar* 13,20 nigin [2,5]7,46,40 *ta-mar*
14. [2,57,4]6,40 *a-na* 4[4...]26,40 daḫ
15. [47,24,2]6,40 *t*[*a-mar*] *mi-na* íb-si
16. [53,20 íb-si 13,20] *ta-ki-il-ta-ka*
17. [*i-na* 53,20 zi 40 *ta-mar* igi 1],20 *šà ki-ma* a-šà gar
18. [*pu-ṭú-úr* 45 *ta-mar* 45 *a-na* 40] *i-ší-ma*
19. [30 *ta-mar* uš *a-pu-sà-mi-ki* 30 *a-na* 1,20 dal]
20. [*šà a-pu-sà-mi-ki i-ší-ma* 40 *ta-mar* 40 dal]
21. [10 *mi-is-si*$_{20}$*-ta a-na* 40 daḫ 50 *ta-mar* 50 nigin]

Reverse

1. [... ... ...] *ta-mar*
2. [... ... ... 41,40] *ta-mar mi-na* íb-si
3. [50 íb-si 10 *ta-ki-il-t*]*a-ka i-na* 50 zi
4. [40 *ta-mar* igi 1,20 *pu*]*-ṭú-úr* 45 *ta-mar*
5. [45 *a-na* 40 *i*]*-ší-ma* 30 *ta-mar* uš *a-pu-sà-mi-ki*
6. [30 *a-na* 1,20] dal *šà a-pu-sà-mi-ik-ki i-ší-ma*
7. 40 *ta-mar* 40 dal 10 *mi-is-si*$_{20}$*-ta šà* uš
8. *a-na* 40 daḫ 50 *ta-mar* uš 5 *mi-*⟨*is*⟩*-si*$_{10}$*-ta*
9. *šà* sag *a-na* 40 daḫ 45 *ta-mar* sag

[6] My transliteration of the first four lines differs from that of Bruins and Rutten, which seems to be incorrect.

Translation

Obverse

1. [... ...]. I go away by [5] (nindan), and [... ...]
2. [... ...] A regular concave-sided tetragon is enclosed(?) therein with the sides (of the inner square).
3. [35,0] is the area in between. What is my side of the (outer) square?
4. [You], multiply 5, the distance (between the outer and inner squares), by 2, and you see 10.
5. Square [10], (and) you see 1,40. Subtract 1,40 from 35,0, (and) you see 33,20.
6. Put down 1 for the regular concave-sided tetragon. Put down 1;20, the diagonal of the regular concave-sided tetragon.
7. Multiply 1;20 by 1, and you see 1;20. Put down 1;20 like the area (of the inner square).
8. Next (literally: return!), square 1;20, (and) you see 1;46,40. Square 1, (and) you see 1.
9. Multiply 1 by 0;26,40, the constant of a regular concave-sided tetragon, and
10. you see 0;26,40. Subtract 0;26,40 from 1;46,40,
11. (and) you see 1;20. Multiply 1;20 by 33,20, the (reduced) area in between, and
12. you see [4]4,26;40. Multiply 1;[20] by 10 of the distance, and
13. you see [13;20]. Square 13;20, (and) you see [2,5]7;46,40.
14. Add [2,57;4]6,40 to [44],26;40,
15. (and) yo[u see 47,24;2]6,40. What is the square root?

16, 17. [53;20 is the square root. Subtract 13;20], your prepared number, [from 53;20, (and) you see 40. The reciprocal of 1];20, {which you put down like the area},

18. [make, (and) you see 0;45]. Multiply [0;45 by 40], and

19 – 21. [you see 30. (This is) the length of the regular concave-sided tetragon. Multiply 30 by 1;20, the diagonal of the regular concave-sided tetragon, and you see 40. 40 is the diagonal. Add 10, the distance, to 40, (and) you see 50. 50 is the side of the square].

Reverse

1. [... ... ...] you see.
2. [... ... ...] you see [41,40]. What is the square root?
3. [50 is the square root. Subtract [10], your [prepared number], from 50, (and)
4. [you see 40. M]ake [the reciprocal of 1;20], (and) you see 0;45.
5. [Mul]tiply [0;45 by 40], and you see 30. (This is) the length of the regular concave-sided tetragon.
6. Multiply [30 by 1;20], the diagonal of the regular concave-sided tetragon, and

7. you see 40. 40 is the diagonal. 10, the distance of the length,

8, 9. add to 40, (and) you see 50. (This is) the length. 5, the distance of the width, add to 40, (and) you see 45. (This is) the width.

Commentary

We cannot completely restore the statement of the first problem, because we have no other problem which has the same mathematical contents. It is possible, however, to reconstruct the quadratic equation solved in it:

$$(1;20x + 2 \cdot 5)^2 - 0;26,40x^2 = 35,0 \tag{1}$$

where x is the length of a quadrant, which is called "the length of the *apsamikkum*" in the text. The quantity asked for is not x but $1;20x + 10$, the length of a side of a square containing the *apsamikkum*. Before discussing the geometrical significance of equation (1), we had better examine the details of the solution.

$$\begin{aligned} 35,0 - (2 \cdot 5)^2 &= 35,0 - 10^2 \\ &= 35,0 - 1,40 \\ &= 33,20 \quad \text{(lines 4 and 5).} \end{aligned}$$

After giving the directions in line 6, the text calculates:

$$\begin{aligned} (1;20 \cdot 1)^2 &= (1;20)^2 \\ &= 1;46,40, \\ 1^2 \cdot 0;26,40 &= 1 \cdot 0;26,40 \\ &= 0;26,40, \\ 1;46,40 - 0;26,40 &= 1;20 \quad \text{(lines 7–11).} \end{aligned}$$

Therefore equation (1) becomes:

$$1;20x^2 + 2 \cdot 10 \cdot 1;20x = 33,20.$$

Multiplying both sides by 1;20, the text obtains:

$$(1;20x)^2 + 2 \cdot 13;20 \cdot 1;20x = 44,26;40 \quad \text{(lines 11–13),}$$

and adding $13;20^2$ ($= 2,57;46,40$) to the results,

$$(1;20x + 13;20)^2 = 47,24;26,40 \quad \text{(lines 13–15).}$$

$$\begin{aligned}
\therefore 1;20x + 13;20 &= \sqrt{47,24;26,40} \\
&= 53;20 \quad \text{(lines 15 and 16)}. \\
\therefore 1;20x &= 40 \quad \text{(lines 16 and 17)} \qquad (2) \\
\therefore x &= \overline{1;20} \cdot 40 \\
&= 0;45 \cdot 40 \\
&= 30 \quad \text{(lines 17–19)}. \\
\therefore 1;20x + 10 &= 1;20 \cdot 30 + 10 \\
&= 40 + 10 \\
&= 50 \quad \text{(lines 19–21)}.
\end{aligned}$$

It should be noted that the coefficient 1;20 of equation (2) is the result of a subtraction in lines 10 and 11, and not the one mentioned in line 7.

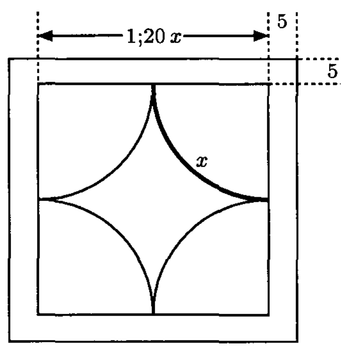


Figure 3

Let us return to equation (1). The geometric figures treated in this problem are probably two squares and one *apsamikkum*. The distance between the outer and the inner squares is 5 nindan ($\approx$ 30m), and the *apsamikkum* is inscribed in the inner square. The area of the *apsamikkum* is subtracted from the area of the outer square, and the result, which is called "the area in between", is 35,0 nindan^2 (cf. Fig. 3). This must be the significance equation (1) implies. Moreover, the number, $2 \cdot 5$, in equation (1), whose square is equal to the total area of the four small squares made by the outer and the inner squares at the corners, may suggest that in Babylonian mathematics the identity,

$$(a + 2b)^2 = a^2 + 4ab + 4b^2,$$

was obtained by means of "geometrical algebra" (cf. Fig. 4).

As to the second problem, the equation and its solution may be reconstructed as follows (cf. Fig. 5):

$$(1;20x + 2 \cdot 5)(1;20x + 5) - 0;26,40x^2 = 30,50.$$

$$(1;20^2 - 0;26,40)x^2 + 15 \cdot 1;20x = 30,50 - 50.$$

$$1;20x^2 + 2 \cdot 7;30 \cdot 1;20x = 30,0.$$

$$(1;20x)^2 + 2 \cdot 10 \cdot 1;20x = 40,0.$$

$$(1;20x + 10)^2 = 41,40 \quad \text{(Reverse, line 2).}$$

$$1;20x + 10 = 50 \quad \text{(line 3).}$$

$$1;20x = 40 \quad \text{(lines 3 and 4).}$$

$$x = 1;20 \cdot 40 = 0;45 \cdot 40 = 30 \quad \text{(lines 4 and 5).}$$

$$1;20x = 1;20 \cdot 30 = 40 \quad \text{(lines 6 and 7).}$$

$$10 + 40 = 50 \quad \text{(lines 7 and 8).}$$

$$5 + 40 = 45 \quad \text{(lines 8 and 9).}$$

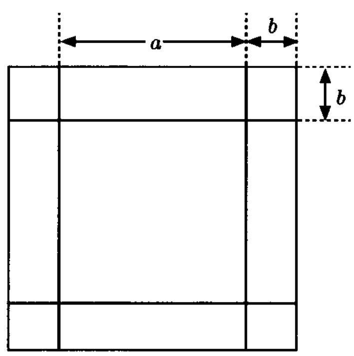


Figure 4

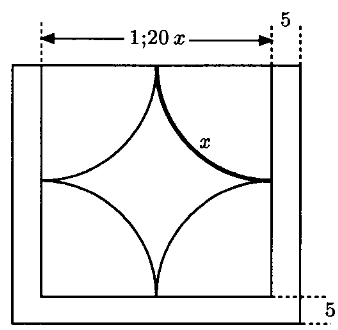


Figure 5

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